Math 181 Spring 2012 Name:

TEST #2 @ 200 points

Write neatly. Show all work. No work, no credit given. Write all responses on separate paper. Clearly label the exercises. No graphing calculator allowed. No phone allowed.

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1. Which sequences (a_n) converge and which diverge? Find the limit of each convergent sequence.

   a. $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$ b. $a_n = (-1)^n\left(1-\dfrac{1}{n}\right)$ c. $a_n = \dfrac{\sin^2 n}{2^n}$

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2. Which series converges , and which diverges? If a series converges, find its sum.

   a. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right)$ b. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{\sqrt{5}}\right)^n$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{6}{(2n-1)(2n+1)}$

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3. a. Find the values of x for which the series converges. Also, find the sum of the series for those values of x.

   $$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^n .$$

   b. For what values of x does the series converge?

   $$\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

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4. Which of the series converge and which diverge? Name the test you are using .

   a. $\displaystyle\sum_{n=1}^{\infty}\ln\frac{1}{3^n}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{e^{n/3}}$ c. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\ln n}$ d. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$

   e. $\displaystyle\sum_{n=0}^{\infty}\frac{e^n}{e^n+n}$ f. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ g. $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{1+e^{2n}}$ h. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n$

   i. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ j. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$ k. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+1}$ l. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$

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5. a) Approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^5}$ by using the sum of the first 4 terms. Estimate the error involved in this approximation.

   b) How many terms are required to ensure that the sum is accurate to within 0.0001?

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6 a. Let $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. Is the series convergent? Do not just write an answer.

   b. Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$.

7. Find the Taylor series generated by $f(x)=\dfrac{1}{x}$ at $a=2$. For what values of x does the series converge?

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8. Find the Maclaurin series for $f(x)=e^{-x}$ using the direct method (the definition of a Maclaurin series). Find its interval of convergence.

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9. Using known series, find the first four nonzero terms of the Maclaurin series for each function and the values of x for which the series converge absolutely.

   a. $f(x)=e^{\sin x}$ b. $g(x)=e^{x}\sin x$

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10. Find the first four terms of the binomial series for $(1-x)^{-\frac{1}{2}}$.

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11. Use series to approximate the values of the integral

   $\int_0^{0.1}\sqrt{1+x^4}\,dx$ with an error less than 10^{-8}

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12. Complete the following definitions, theorems, or properties:

   a. If a sequence (a_n) is bounded and monotonic, then the sequence ______________________________.

   b. If $\lim_{n\to\infty}|a_n|=0$, then ____

   c. If a series $\sum a_n$ is convergent, then (a_n) ______________________________________

   d. If $\sum|a_n|$ is convergent, then ____

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13. What are the power representations of the following functions? What is the radius of convergence of each? <u>Do not prove.</u>

   a. $\dfrac{1}{1-x}=$ b. $e^{x}=$

   c. $\sin x=$ d. $\cos x=$

Extra Credit

1. (3 points) Prove the following theorem:

   A series of nonnegative terms converges if and only if the sequence of its partial sums is bounded from above.

2. (3 points) First, state the Direct Comparison Theorem, then prove it.

3. (4 points) For which positive integers k is the following series convergent $\sum_{n=1}^{\infty}\dfrac{(n!)^2}{(kn)!}$?

MATH 181

TEST 2 - SOLUTIONS

① (a) $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}, \ n \geq 1$

A sequence (a_n) is convergent iff $\lim_{n\to\infty} a_n \in \mathbb{R}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} \quad \left(\frac{\infty}{\infty}\text{-case. use l'Hopital}\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{n}+\frac{1}{\sqrt{n}}} = \frac{2}{\infty} = 0$$

Therefore, (a_n) is convergent and $\boxed{\lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} = 0}$

(b) $a_n = (-1)^n\left(1-\frac{1}{n}\right), \ n \geq 1$

$$\lim_{n\to\infty} a_n = \begin{cases} 1 & \text{if } n = \text{even} \\ -1 & \text{if } n = \text{odd} \end{cases}$$

$\left(\text{as } \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1\right)$

therefore, $\lim_{n\to\infty} a_n$ does not exist

so $\boxed{(a_n) \text{ is divergent}}$

(c) $a_n = \dfrac{\sin^2 n}{2^n}$

$$\left.\begin{array}{l} 0 \leq a_n = \dfrac{\sin^2 n}{2^n} \leq \dfrac{1}{2^n} \\ \lim_{n\to\infty} 0 = \lim_{n\to\infty} \dfrac{1}{2^n} = 0 \end{array}\right\} \xRightarrow{\text{sandwich theorem}}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0}$$

so (a_n) is convergent.

② (a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$

A series $\sum_{n=1}^{\infty} a_n$ is convergent iff $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \in \mathbb{R}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\left(\cancel{\frac{1}{\ln 3}} - \frac{1}{\ln 2}\right) + \left(\cancel{\frac{1}{\ln 4}} - \cancel{\frac{1}{\ln 3}}\right) + \dots + \left(\cancel{\frac{1}{\ln(n+1)}} - \cancel{\frac{1}{\ln n}}\right) + \left(\frac{1}{\ln(n+2)} - \cancel{\frac{1}{\ln(n+1)}}\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{\ln(n+2)} - \frac{1}{\ln 2}\right) = \frac{1}{\infty} - \frac{1}{\ln 2}$$

$$= 0 - \frac{1}{\ln 2} = -\frac{1}{\ln 2}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$ is convergent and

$$\boxed{\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right) = -\frac{1}{\ln 2}}$$

(b) $\sum_{n=0}^{\infty} \left(\frac{2}{\sqrt{5}}\right)^n$ is a geometric series with first term $a = \left(\frac{2}{\sqrt{5}}\right)^0 = 1$

and ratio $r = \frac{2}{\sqrt{5}} \in (-1,1)$

Therefore, the series is convergent and its sum is $\frac{a}{1-r} = \frac{1}{1-\frac{2}{\sqrt{5}}} = \frac{\sqrt{5}}{\sqrt{5}-2}$

$$\boxed{\sum_{n=0}^{\infty} \left(\frac{2}{\sqrt{5}}\right)^n = \frac{\sqrt{5}}{\sqrt{5}-2}}$$

(c) $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

$$\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$6 = A(2n+1) + B(2n-1)$$

$$6 = (2A+2B)n + (A-B)$$

$$\begin{cases} 2A+2B = 0 \\ A-B = 6 \end{cases} \Rightarrow \begin{cases} A+B = 0 \\ A-B = 6 \end{cases} \Rightarrow$$

$2A = 6 \Rightarrow A = 3$ and $B = -3$

So $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty}\left(\frac{3}{2n-1} - \frac{3}{2n+1}\right)$

we'll find s_n and $\lim_{n\to\infty} s_n$

$$s_n = \sum_{k=1}^{n}\left(\frac{3}{2k-1} - \frac{3}{2k+1}\right) = 3\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

$$= 3\left(\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \dots\right.$$

$$\left. + \dots + \left(\cancel{\frac{1}{2(n-1)-1}} - \cancel{\frac{1}{2(n-1)+1}}\right) + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)\right]$$

$$= 3\left(1 - \frac{1}{2n+1}\right), \quad \lim s_n = 3(1-0) = 3$$

Therefore, $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$ is conv.

and $\boxed{\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = 3}$

3 (a) $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^n = \sum_{n=0}^{\infty}\left(-\frac{x-3}{2}\right)^n$

is a geometric series with first term $a=1$ and ratio $r = -\frac{x-3}{2}$

The series converges when

$$|r| = \left|-\frac{x-3}{2}\right| = \frac{|x-3|}{2} < 1 \iff$$

$$|x-3| < 2 \iff$$

$$-2 < x-3 < 2 \iff$$

$$1 < x < 5$$

The series diverges when $|r| \geq 1$, when $r \leq 1$ or $r \geq 5$

Therefore, $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^n$ converges when $x \in (1,5)$

and its sum is

$$\frac{a}{1-r} = \frac{1}{1-\left(-\frac{x-3}{2}\right)} = \frac{2}{x-1}$$

$$\boxed{\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^2 = \frac{2}{x-1}, \ \forall x \in (1,5)}$$

(6) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

we'll apply the Ratio Test to

$\sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right|$; let $a_n = \frac{x^n}{n!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$

$= \lim_{n\to\infty} \frac{|x|}{(n+1)} = |x| \lim_{n\to\infty} \frac{1}{n+1}$

$= |x| \cdot 0 = 0, \ \forall\, x \in \mathbb{R}$

$\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is convergent } \forall\, x \in \mathbb{R}}$

(4) (a) $\sum_{n=1}^{\infty} \ln\left(\frac{1}{3^n}\right)$

Note that $\lim_{n\to\infty} \left(\ln \frac{1}{3^n}\right) =$

$= \lim_{n\to\infty} (\ln 1 - \ln 3^n)$

$= \lim_{n\to\infty} (0 - n \ln 3) = -\infty$

By the n^{th}-term test for divergence,

$\boxed{\sum_{n=1}^{\infty} \ln\left(\frac{1}{3^n}\right) \text{ is divergent}}$

(b) $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$; let $a_n = \frac{n^2}{e^{n/3}}$

$a_n > 0, \ \forall\, n \geq 1$

we'll apply the Ratio Test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(n+1)^2}{e^{\frac{n+1}{3}}} \cdot \frac{e^{\frac{n}{3}}}{n^2}\right]$

$= \lim_{n\to\infty} \frac{1}{e^{1/3}} \cdot \frac{(n+1)^2}{n^2} = \frac{1}{e^{1/3}} \cdot 1 < 1$

so $\boxed{\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}} \text{ is convergent}}$.

(c) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$; let $a_n = \frac{1}{\sqrt{n}\ln n}, n \geq 2$

$a_n > 0, \ \forall\, n \geq 2$

let $b_n = \frac{1}{n}$, $b_n > 0$ with

$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$ divergent

(p-series with $p=1$)

We'll apply the Limit Comparison Test

$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}\ln n}}$

$= \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}}$ $\left(\frac{\infty}{\infty}\text{-case, use l'Hopital}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$

Therefore, $\sum a_n$ and $\sum b_n$ both converge or both diverge; but $\sum b_n$ is div.

so $\boxed{\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n} \text{ is divergent}}$

(d) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

$\frac{\ln n}{n^3} \geq 0 \quad \forall n \geq 1$

we'll use the Direct Comparison Test

$0 \leq \frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}, \quad \forall n \geq 1$

with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent

(as a p-series with $p=2$)

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{\ln n}{n^3} \text{ is convergent}}$

(e) $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$

Note that $\lim_{n\to\infty} \frac{e^n}{e^n+n} = \left(\frac{\infty}{\infty}\text{-case use l'Hopital}\right)$

$= \lim_{n\to\infty} \frac{e^n}{e^n+1} \left(\frac{\infty}{\infty}\text{-case use l'Hopital}\right)$

$= \lim_{n\to\infty} \frac{e^n}{e^n} = \lim_{n\to\infty} 1 = 1 \neq 0$

By the n^{th} term test for divergence,

$\boxed{\sum_{n=0}^{\infty} \frac{e^n}{e^n+n} \text{ is divergent}}$

(f) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ is an alternating series

let $a_n = \frac{1}{\sqrt{n}}$

- $a_n > 0 \quad \forall n \geq 1$
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$
- $a_n = \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}} = a_{n+1}$

so $a_n \geq a_{n+1} \quad \forall n \geq 1$

By the Alternating Series Test

$\Rightarrow \boxed{\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \text{ is convergent}}$

(g) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$; let $a_n = \frac{e^n}{1+e^{2n}}$

$a_n > 0 \quad \forall n \geq 1$

We'll apply the Ratio Test

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{n+1}}{1+e^{2(n+1)}} \cdot \frac{1+e^{2n}}{e^n}$

$= \lim_{n\to\infty} \frac{e(1+e^{2n})}{1+e^{2n+2}} \left(\frac{\infty}{\infty}\text{-case use l'Hopital}\right)$

$= e \lim_{n\to\infty} \frac{2e^{2n}}{2e^{2n+2}} = e \lim_{n\to\infty} \frac{1}{e^2}$

$= e \cdot \frac{1}{e^2} = \frac{1}{e} < 1$

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}} \text{ is conv.}}$

(h) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$; let $a_n = \left(\frac{2n+3}{5n+4}\right)^n$

$a_n > 0, \quad \forall n \geq 1$

we'll apply the Root Test

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2n+3}{5n+4} = \frac{2}{5} < 1$

therefore, $\boxed{\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n \text{ is conv.}}$

(i) $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$; let $a_n=\frac{(n!)^2}{(2n)!}$

$a_n>0$, $\forall n\geq 1$

we'll apply the Ratio Test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{((n+1)!)^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2}$$

$$=\lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)}=\frac{1}{4}<1$$

so $\boxed{\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!} \text{ is convergent}}$

(j) $\sum_{n=1}^{\infty}\frac{\ln n}{n}$; let $a_n=\frac{\ln n}{n}$

$a_n>0$, $\forall n\geq 2$

$\frac{\ln n}{n}>\frac{1}{n}$, $\forall n\geq 2$

and $\sum_{n=2}^{\infty}\frac{1}{n}$ is divergent

(p-series with $p=1$)

By Direct Comparison Test,

$\sum_{n=2}^{\infty}\frac{\ln n}{n}$ is divergent,

therefore

$\boxed{\sum_{n=1}^{\infty}\frac{\ln n}{n} \text{ is divergent}}$

(k) $\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+1}$ is an alternating series

let $a_n=\frac{n^2}{n^3+1}$, $n\geq 1$

- $a_n>0$, $\forall n\geq 1$
- $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n^2}{n^3+1}=0$
- prove $a_n\geq a_{n+1}$

let $f(x)=\frac{x^2}{x^3+1}>0$, $\forall x\geq 1$

$f(n)=a_n$, $\forall n\geq 1$, $n\in\mathbb{N}$

$$f'(x)=\frac{2x(x^3+1)-3x^2(x^2)}{(x^3+1)^2}=\frac{2x-x^4}{(x^3+1)^2}$$

$$f'(x)=\frac{x(2-x^3)}{(x^3+1)^2}$$

$f'(x)<0$ iff $2-x^3<0$

iff $x^3>2$

iff $x>\sqrt[3]{2}\simeq 1.2$

so $f(x)$ is decreasing $\forall x\geq 2$

so $a_n\geq a_{n+1}$ $\forall n\geq 2$

By the Alternating Series Test $\Rightarrow$

$\boxed{\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+1} \text{ is convergent}}$

(e) $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

$0 \le \left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}, \quad \forall n \ge 1$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

(p-series with $p=2>1$)

by Direct Comparison Test

$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is convergent,

therefore $\boxed{\sum_{n=1}^{\infty} \frac{\cos n}{n^2} \text{ is conv.}}$

(5) (a) $\sum_{n=1}^{\infty} \frac{2}{n^5}$ is convergent

(p-series with $p=5>1$)

so there is $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{k^5} = S$

and $\sum_{n=1}^{\infty} \frac{2}{n^5} = S$

Therefore, $\sum_{n=1}^{\infty} \frac{2}{n^5} \approx S_4$

$$\boxed{\sum_{n=1}^{\infty} \frac{2}{n^5} \approx \frac{2}{1} + \frac{2}{2^5} + \frac{2}{3^5} + \frac{2}{4^5} \approx 2.07}$$

By the Remainder Estimate for the integral Test,

$R_n \le \int_n^{\infty} f(x)\,dx$

(Note that $f(x) = \frac{2}{x^5} > 0 \ \forall x \ge 1$

and $f(x)$ is decreasing $\forall x \ge 1$)

so $R_4 \le \int_4^{\infty} \frac{2}{x^5}\,dx = \lim_{t\to\infty} \int_4^t 2x^{-5}\,dx$

$= \lim_{t\to\infty} 2\int_4^t x^{-5}\,dx$

$= \lim_{t\to\infty} \left.\frac{2x^{-4}}{-4}\right]_4^t$

$= \lim_{t\to\infty} -\frac{1}{2}\left(\frac{1}{t^4} - \frac{1}{4^4}\right) = -\frac{1}{2}\left(-\frac{1}{4^4}\right)$

≈ 0.00195

$\therefore \boxed{\text{Error} \le 0.00195}$

(b) $n = ?$ such that $R_n \le 0.0001$

let $\int_n^{\infty} \frac{2}{x^5}\,dx \le 0.0001$

$\lim_{t\to\infty} \int_n^t 2x^{-5}\,dx = \lim_{t\to\infty} \left.\frac{2x^{-4}}{-4}\right]_n^t$

$= -\frac{1}{2} \lim_{t\to\infty} \left(\frac{1}{t^4} - \frac{1}{n^4}\right) = -\frac{1}{2}\left(-\frac{1}{n^4}\right)$

$= \frac{1}{2n^4}$

so $\frac{1}{2n^4} \le 0.0001$

$2n^4 \ge 10^4$

$n^4 \ge \frac{10^4}{2}$, so $n \ge \sqrt{\frac{10^4}{2}} \approx 70.7$

so we need $\boxed{n = 71}$

(6) (a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is an alternating series

let $\begin{cases} a_n = \frac{1}{n} > 0, \forall n \geq 1 \\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \\ a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1} \\ \text{so } a_n > a_{n+1} \quad \forall n \geq 1 \end{cases}$

By the Alternating Series Test

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent

(b) $|\text{error}| \leq \left|(-1)^{5+1} \frac{1}{5}\right| = \frac{1}{5} = 0.2$

$\boxed{|\text{error}| \leq 0.2}$

(7) $f(x) = \frac{1}{x}$, $a = 2$

The Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n$$

$f(x) = \frac{1}{x} \qquad f(2) = \frac{1}{2}$

$f'(x) = -x^{-2} \quad f'(2) = -2^{-2}$

$f''(x) = (-1)(-2) x^{-3} \quad f''(2) = 2!\, 2^{-3}$

$f'''(x) = (-1)(-2)(-3) x^{-4} \quad f'''(2) = -3!\, 2^{-4}$

so $f^{(n)}(x) = (-1)^n x^{-(n+1)}$

$f^{(n)}(2) = (-1)^n \frac{n!}{2^{n+1}}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{n!}{n!\, 2^{n+1}} (x-2)^n =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \left(-\frac{x-2}{2}\right)^n$$

This is a geometric series of ratio $r = -\frac{x-2}{2}$

It is convergent when

$|r| = \left|-\frac{x-2}{2}\right| = \frac{|x-2|}{2} < 1$

iff $|x-2| < 2$

$-2 < x-2 < 2$

$0 < x < 4$

and it is divergent when $|r| \geq 1$, when $x \leq 0$ or $x \geq 4$

so $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n$ is conv. when $x \in (0,4)$

(8) $f(x) = e^{-x}$,

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f'(x) = -e^{-x} \qquad f'(0) = -1$

$f''(x) = e^{-x} \qquad f''(0) = 1$

$f^{(n)}(x) = (-1)^n$ so

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

Apply the Ratio Test to $\sum_{n=0}^{\infty} |a_n|$, $a_n = \frac{(-1)^n x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{n+1} = 0, \quad \forall x\in\mathbb{R}$$

so $\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n \text{ converges } \forall x\in\mathbb{R}}$

(9) (a) $f(x) = e^{\sin x}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \ \forall x$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \ \forall x$

Then,

$e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \frac{(\sin x)^3}{3!} + \cdots, \ \forall x$

$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) +$

$+ \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{3!} + \cdots\right)^3 + \cdots \ \forall x$

$= 1 + x - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} + \cdots + \frac{1}{2}\left(x^2 - \frac{2}{6}x^4 + \cdots\right)$

$+ \cancel{\frac{1}{6}x^3} + \cdots$

$= \boxed{1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^4 + \cdots \quad \forall x}$

(b) $e^x \sin x =$

$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$

$= x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$

$+ x^2 - \cancel{\frac{x^4}{6}} + \cdots$

$+ \frac{x^3}{2} - \frac{x^5}{12} + \cdots$

$+ \cancel{\frac{x^4}{6}} + \cdots = x + x^2 + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 +$

$+ \frac{x^5}{24} \qquad + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5$

so $\boxed{e^x \sin x = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots \quad \forall x}$

(10) $(1-x)^{-\frac{1}{2}} = \left(1 + (-x)\right)^{-\frac{1}{2}}$

$= 1 + \sum_{n=1}^{\infty}\binom{-\frac{1}{2}}{n}(-x)^n, \quad \forall |x|<1$

$= 1 + \binom{-\frac{1}{2}}{1}(-x) + \binom{-\frac{1}{2}}{2}(-x)^2 +$

$+ \binom{-\frac{1}{2}}{3}(-x)^3 + \binom{-\frac{1}{2}}{4}(-x)^4 + \cdots$

$= 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 +$

$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\cdot 2\cdot 3}(-x^3) +$

$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$

$\boxed{(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots \quad \forall |x|<1}$

(11) $\sqrt{1+x^4} = (1+x^4)^{\frac{1}{2}} =$

$= 1 + \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(x^4)^n, \quad \forall |x|<1$

$= 1 + \frac{\frac{1}{2}}{1}x^4 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(x^4)^2 +$

$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot 2\cdot 3}(x^4)^3 +$

$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\cdot 2\cdot 3\cdot 4}(x^4)^4 + \cdots$

$= 1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \cdots$

Then,

$$\int_0^{0.1} \sqrt{1+x^4}\, dx =$$

$$= \int_0^{0.1} \left(1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \cdots\right) dx$$

$$= \left[x + \frac{x^5}{2\cdot 5} - \frac{x^9}{8\cdot 9} + \frac{x^{13}}{16\cdot 13} - \cdots\right]_0^{0.1}$$

$$= 0.1 + \frac{(0.1)^5}{10} - \frac{(0.1)^9}{72} + \frac{(0.1)^{13}}{16\cdot 13} - \cdots$$

$$\approx 0.1 + \frac{(0.1)^5}{10} = 0.100001$$

where $|\text{error}| \le \frac{(0.1)^9}{72} \approx 1.39 \times 10^{-11}$

(12) (a) If (a_n) is bounded and monotonic, then the sequence $\boxed{(a_n) \text{ is convergent}}$

(b) if $\lim_{n\to\infty} |a_n| = 0$, then

$\boxed{\lim_{n\to\infty} a_n = 0}$

(c) If $\sum a_n$ is convergent, then (a_n) has limit zero, that is $\boxed{\lim_{n\to\infty} a_n = 0}$

(d) If $\sum |a_n|$ is convergent, then $\boxed{\sum a_n \text{ is convergent}}$

(13)

(a) $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$

$\forall\ |x| < 1$, so $R = 1$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\forall\ x$, so $R = \infty$

(c) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\forall\ x$, so $R = \infty$

(d) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, $\forall x$

so $R = \infty$

EXTRA CREDIT

(1) and (2) - see notes or textbook for proofs of the theorems

(3) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$; let $a_n = \frac{(n!)^2}{(kn)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!^2}{(k(n+1))!} \cdot \frac{(kn)!}{(n!)^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(kn+1)\cdots(kn+k)}$$

Note that the denominator has k factors.

If $(k=1)$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n+1} = \infty$, so the series diverges

If $(k=2)$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}$, so the series converges

If $(k>2)$, then the highest power of n in the denominator is greater than 2, so the limit is 0, so the series converges

therefore, the series converges when $\underline{\underline{k \geq 2}}$