Math 181 Spring 2012 Name:

TEST #1 @ 200 points

Write neatly. Show all work. No work, no credit given. Write all responses on separate paper. Clearly label the exercises. No graphing calculator allowed. No phone allowed.

1. A region is bounded by $y = x$, $y = 1$, and $x = 0$.

   a) Find the volume of the solid generated by revolving the region about $x = -1$.

   b) If we fill the vase formed in part (a) with water at a constant rate of 2 cubic units per second, how fast will the water level in the bowl be rising when the water is 0.3 units deep?

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2. Find the volume of the solid generated by revolving the region enclosed by the graphs of $y = e^{\frac{x}{2}}, y = 1$, and $x = \ln 3$ about the x-axis.

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3. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all the water 2 m above the top of the tank. (The weight - density of water is 9800 N/m^3).

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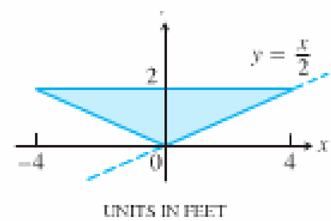
4. The vertical triangular plate shown here is the end plate of a trough full of water. Find the fluid force against the plate . The weight-density of freshwater is 64 lb/cubic feet.



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5. Prove the following formulas:

   a) $\cosh 2x = \cosh^2 x + \sinh^2 x$ b) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

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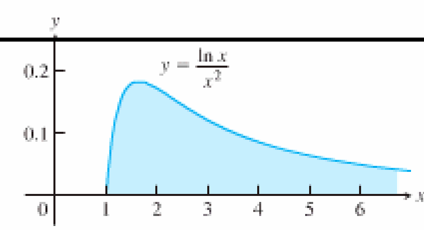
6. Is the area under the curve $y = \frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite?

   If so, what is its value?



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7. If f' is continuous on $[a,b]$, find the length of the curve $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$.

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8. Solve the initial value problem for x in terms of t: $(t^2 - 3t + 2)\frac{dx}{dt} = 1$, $t > 2, x(3) = 0$.

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9. Find the following integrals:

a) $\int_0^{\sqrt{3}/2} \frac{4x^2 dx}{\left(1-x^2\right)^{3/2}}$ b) $\int \sin^4 q\, dq$ c) $\int_{-\infty}^{\infty} 2xe^{-x^2}\, dx$

d) $\int_1^3 \frac{2}{\sqrt{x-1}} dx$ e) $\int_0^1 \ln x dx$ f) $\int_0^{p/2} \cos^2 y dy$

g) $\int \frac{x^3}{\sqrt{x^2+4}} dx$ h) $\int \frac{dx}{x\left(x^2+1\right)^2}$ i) $\int_{\frac{5p}{6}}^{p} \frac{\cos^4 x}{\sqrt{1-\sin x}} dx$

j) $\int \sec^3 q\, dq$ k) $\int_{-\infty}^{0} xe^x dx$ l) $\int \sin 2t \cos 4t dt$

m) $\int \frac{dx}{x^2+x+3}$ n) $\int \sin^{-1} x dx$

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10. Use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence:

a) $\int_0^2 \frac{dx}{1-x^2}$ b) $\int_1^{\infty} \frac{dx}{x^5+2}$ c) $\int_0^{\infty} \sin x dx$

EXTRA CREDIT

Exercise #1 @ 1 point Prove that if f is a continuous function, then $\int_0^a f(x) = \int_0^a f(a-x)\,dx$.

Exercise #2 @ 4 points Approximate the definite integral of a continuous function on the interval $[a,b]$ using parabolas; that is, prove Simpson's Rule used to approximate $\int_a^b f(x)\,dx$.

Exercise #3 @ 3 points Find $\int \frac{x^2+1}{x\sqrt{x^4+1}} dx$, $x > 0$.

Exercise #4 @ 4 points Find the value of the constant C for which the integral $\int_0^{\infty} \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx$ converges. Evaluate the integral for this value of *C*.

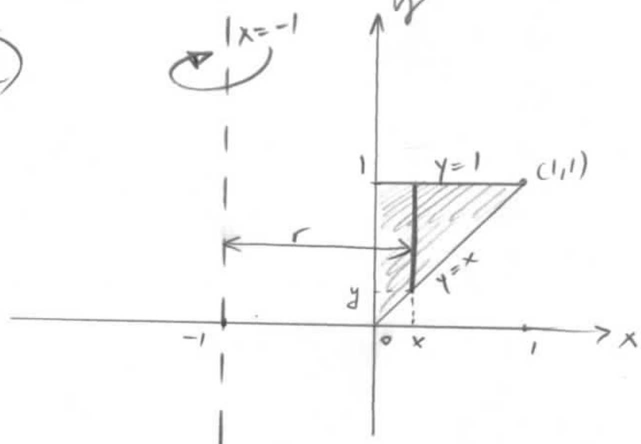
Exercise #5 @ 3 points Water in an open bowl evaporates at a rate proportional to the area of the surface of the water. (This means that the rate of decrease of the volume is proportional to the area of the surface.) Show that the depth of the water decreases at a constant rate, regardless of the shape of the bowl.

MATH181

TEST I - SOLUTIONS

(1) $y=x,\ y=1,\ x=0$

(a)



Shell method - vertical shells from $x=0$ to $x=1$ with

$r = 1+x$,

$h = 1-y = 1-x$

$V = \int_0^1 2\pi (r)(h)\,dx$

$V = 2\pi \int_0^1 (1+x)(1-x)\,dx$

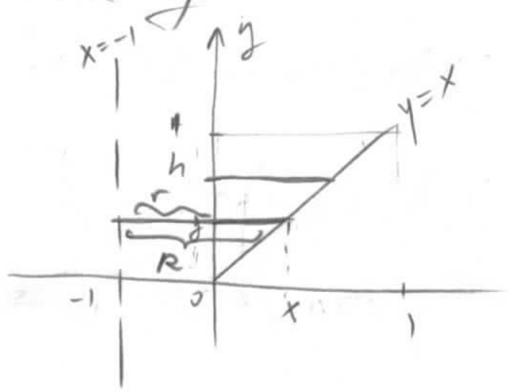
$= 2\pi \int_0^1 (1-x^2)\,dx$

$= 2\pi \left[x - \frac{x^3}{3} \right]_0^1 = 2\pi\left(1-\frac{1}{3}\right) = \frac{4\pi}{3}$

$\boxed{V = \frac{4\pi}{3} \text{ cubic units}}$

(b) let h = height of water in the bowl.

V = volume of water



Let h = height of water

V = vol. of water

use horizontal slabs, ⊥ to y-axis at y from $y=0$ to $y=h$

$\therefore V = \int_0^h A(y)\,dy$

$A(y)$ = washer

$= \pi(R^2 - r^2)$, $R = 1+y$

$r = 1$

$V = \int_0^h \pi\left[(1+y)^2 - 1^2\right] dy$

$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$

$\frac{dV}{dt} = \pi\left[(1+h)^2 - 1\right]\frac{dh}{dt}$

$\frac{dV}{dt} = 2$ cubic units/sec

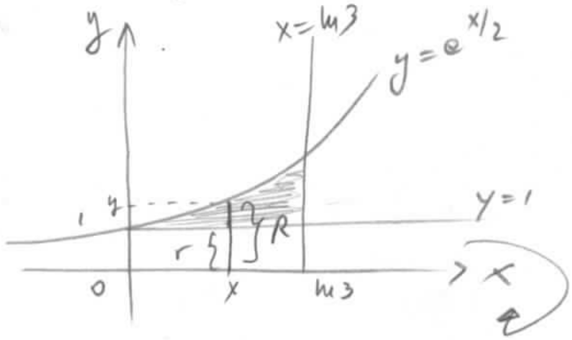
$h = 0.3$ units

find $\frac{dh}{dt}$

$2 = \pi\left(2(0.3) + (0.3)^2\right)\cdot\frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{2}{0.69\pi} \approx 0.92 \text{ units/sec}}$

(2) $y = e^{x/2}$, $y = 1$, $x = \ln 3$



Use vertical planes from $x = 0$ to $\ln 3$

$$V = \int_0^{\ln 3} A(x)\,dx$$

where $A(x) =$ washer

$A(x) = \pi(R^2 - r^2)$, $R = y = e^{x/2}$

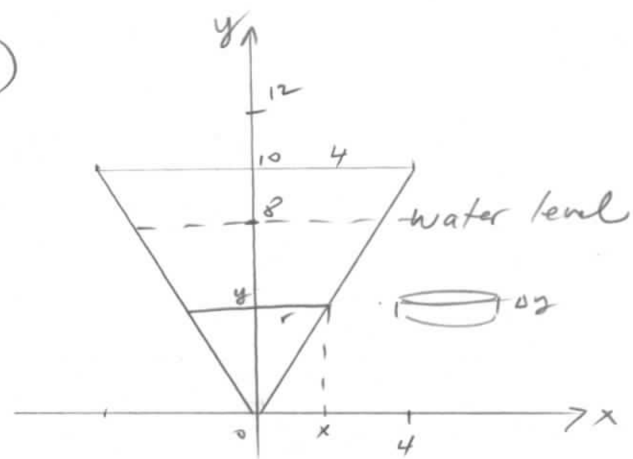
$r = 1$

$$V = \int_0^{\ln 3} \pi(e^x - 1)\,dx$$

$$= \pi\left[e^x - x\right]_0^{\ln 3} = \pi(3 - \ln 3 - 1)$$

$$\boxed{V = (2 - \ln 3)\pi \text{ units}^3}$$

(3)



"Cut" the liquid into n "layers"

Let W_k = work done to pump the k^{th} layer up

Then, $W \approx \sum_{k=1}^{n} W_k$

$W_k = F_k \cdot d_k$

$\begin{cases} F_k = \text{weight of the layer} \\ d_k = \text{distance} \end{cases}$

$F_k = w \cdot V_k$, V_k = volume of layer

$= w(\pi r^2 \Delta y)$

where $\frac{r}{4} = \frac{y}{10} \Rightarrow r = \frac{2y}{5}$

$F_k = 9800\pi \frac{4y^2}{25} \Delta y$

$d_k = 12 - y$

Therefore,

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} \left(9800\pi \frac{4y^2}{25} \Delta y\right)(12 - y)$$

$$W = \int_0^8 9800\pi \cdot \frac{4y^2}{25}(12 - y)\,dy$$

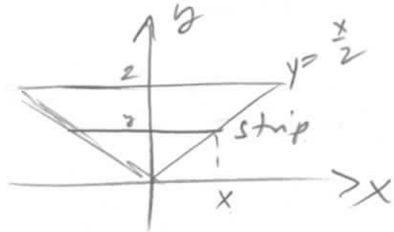
$$= 9800\pi \cdot \frac{4}{25} \int_0^8 (12y^2 - y^3)\,dy$$

$$= 1568\pi\left[4y^3 - \frac{y^4}{4}\right]_0^8$$

$$= 1568\pi\,(1024)$$

$$\boxed{\begin{aligned} W &= 1{,}605{,}632\pi \text{ N} \\ W &\approx 5{,}044{,}241.7 \text{ N} \end{aligned}}$$

(b) $F = w\int_0^2 \begin{pmatrix}\text{strip}\\ \text{length}\end{pmatrix}\begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} dy$



strip length $= 2x = 2(2y) = 4y$
strip depth $= 2 - y$

$F = 64\int_0^2 4y(2-y)\,dy$

$= 64\int_0^2 (8y - 4y^2)\,dy$

$= 64\left(4y^2 - \frac{4}{3}y^3\Big]_0^2\right)$

$= 64\left(16 - \frac{32}{3}\right) = 64\cdot\frac{16}{3}$

$\boxed{F = \frac{1024}{3} \approx 341.3 \text{ lb}\cdot\text{ft}}$

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(5) (a) $\cosh 2x = \cosh^2 x + \sinh^2 x$

Proof

$\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$

Then, $\cosh(2x) = \frac{e^{2x} + e^{-2x}}{2}$

$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$

$= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} - 2 + e^{-2x}}{4}$

$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$

Therefore,
$\cosh 2x = \cosh^2 x + \sinh^2 x$.

(b) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

Proof

$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$

$= \frac{4e^x e^{-x}}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}}\right)^2$

$= \operatorname{sech}^2 x$

so $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

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(6) Area $= \int_1^\infty \frac{\ln x}{x^2}\,dx$

$= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx$

let $\begin{bmatrix} f = \ln x & g' = x^{-2} \\ f' = \frac{1}{x} & g = \frac{-1}{x} \end{bmatrix}$

$= \lim_{t\to\infty}\left(-\frac{1}{x}\ln x\Big]_1^t + \int_1^t \frac{1}{x^2}\,dx\right)$

$= \lim_{t\to\infty}\left(-\frac{1}{t}\ln t - \frac{1}{x}\Big]_1^t\right)$

$$\text{Area} = \lim_{t\to\infty}\left(-\frac{1}{t}\ln t - \frac{1}{t} + 1\right)$$

$$= 1 - \lim_{t\to\infty}\frac{1}{t} - \lim_{t\to\infty}\frac{\ln t}{t} \quad \left(\frac{\infty}{\infty}\right)$$

$$= 1 - 0 - \lim_{t\to\infty}\frac{\frac{1}{t}}{1} = 1$$

so the area is finite

$$\boxed{\text{Area} = 1\ \text{unit}^2}$$

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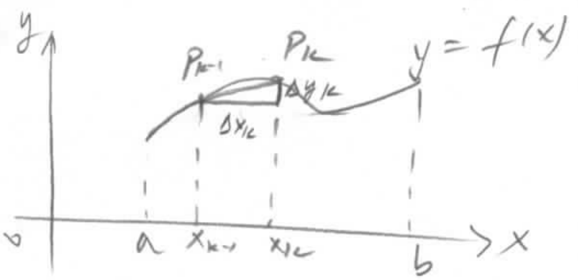
(7)



- partition $[a,b]$

$a = x_0 < x_1 < x_2 < \ldots < x_n = b$

L = length of the curve

$$L = \lim_{n\to\infty}\sum_{k=1}^{n} P_{k-1}P_k$$

$$= \lim_{n\to\infty}\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

but, by the Mean Value Theorem

$\exists\ c_k \in [x_{k-1}, x_k]$ such that

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k}$$

so $\Delta y_k = f'(c_k)\cdot\Delta x_k$

$$L = \lim_{n\to\infty}\sqrt{(\Delta x_k)^2 + (\Delta x_k)^2(f'(c_k))^2}$$

$$= \lim_{n\to\infty}\Delta x_k\sqrt{1 + [f'(c_k)]^2}$$

Therefore,

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx \quad \text{or}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

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(8) $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1$

$t > 2,\quad x(3) = 0$

Solution

$$\frac{dx}{dt} = \frac{1}{t^2 - 3t + 2}$$

$$\frac{dx}{dt} = \frac{1}{(t-1)(t-2)}$$

$$x = \int \frac{1}{(t-1)(t-2)}\,dt$$

$$\frac{1}{(t-1)(t-2)} = \frac{A}{t-1} + \frac{B}{t-2}$$

$1 = A(t-2) + B(t-1)$

$1 = At + Bt - 2A - B$

$$\begin{cases} A + B = 0 \\ -2A - B = 1 \end{cases}$$

$(+)\quad -A = 1 \Rightarrow A = -1$

then $B = 1$

so $\dfrac{1}{(t-1)(t-2)} = \dfrac{-1}{t-1} + \dfrac{1}{t-2}$

$$x = \int \frac{-1}{t-1}\,dt + \int \frac{1}{t-2}\,dt$$

$x = -\ln|t-1| + \ln|t-2| + C$

$t > 2$

$x = -\ln(t-1) + \ln(t-2) + C$

$x(3)=0 \Rightarrow$

$0=-\ln 2 + C \Rightarrow C = \ln 2$

Therefore,

$$\boxed{\begin{aligned} x &= -\ln(t-1)+\ln(t-2)+\ln 2 \\ x &= \ln\frac{2(t-2)}{t-1} \end{aligned}}$$

(9) (a) $I=\int_0^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

let $x=\sin t$, $t\in(-\frac{\pi}{2},\frac{\pi}{2})$

(note $x\neq\pm1$, so $t\neq\pm\frac{\pi}{2}$)

$dx=\cos t\,dt$

$1-x^2=\cos^2 t$

$x=0$, $t=0$

$x=\frac{\sqrt{3}}{2}$, $t=\frac{\pi}{3}$

Then,

$$I=\int_0^{\pi/3}\frac{4\sin^2 t\cdot\cos t\,dt}{(\cos^2 t)^{3/2}}$$

$$=4\int_0^{\pi/3}\frac{\sin^2 t}{\cos^2 t}dt$$

$$=4\int_0^{\pi/3}\frac{1-\cos^2 t}{\cos^2 t}dt$$

$$=4\int_0^{\pi/3}(\sec^2 t-1)\,dt$$

$$=4\left(\tan t\Big]_0^{\pi/3}-t\Big]_0^{\pi/3}\right)$$

$$=4\left(\sqrt{3}-\frac{\pi}{3}\right)$$

so $\boxed{I=4\sqrt{3}-\frac{4}{3}\pi}$

(b) $I=\int\sin^4\theta\,d\theta$

$$=\int(\sin^2\theta)^2\,d\theta$$

$$=\int\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$=\frac{1}{4}\int(1-2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$=\frac{1}{4}\left(\theta-\sin 2\theta+\int\frac{1+\cos 4\theta}{2}d\theta\right)$$

$$=\frac{1}{4}\theta-\frac{1}{4}\sin 2\theta+\frac{1}{8}\left(\theta+\frac{1}{4}\sin 4\theta\right)$$

$$=\frac{1}{4}\theta-\frac{1}{4}\sin 2\theta+\frac{1}{8}\theta+\frac{1}{32}\sin 4\theta+C$$

$$\boxed{I=\frac{3}{8}\theta-\frac{1}{4}\sin 2\theta+\frac{1}{32}\sin 4\theta+C}$$

(c) $I=\int_{-\infty}^{\infty}2xe^{-x^2}dx$

$$I=\int_{-\infty}^{0}2xe^{-x^2}dx+\int_0^{\infty}2xe^{-x^2}dx$$

let $I_2=\int_0^{\infty}2xe^{-x^2}dx$

$$=\lim_{t\to\infty}\int_0^t 2xe^{-x^2}dx$$

let $e^{-x^2}=u$

$-2xe^{-x^2}dx=du$

$x=0$, $u=1$

$x=t$, $u=e^{-t^2}$

$$=\lim_{t\to\infty}\int_0^{e^{-t^2}}-u\,du = -$$

$$=-\lim_{t\to\infty}\frac{u^2}{2}\Big]_0^{e^{-t^2}} = +1$$

Similarly, $I_1=\int_{-\infty}^{0}2xe^{-x^2}dx=-1$

so $\boxed{I=I_1+I_2=0}$

(d) $\int_1^3 \frac{2}{\sqrt{x-1}}\,dx =$

$= \lim_{t\to 1^+} \int_t^3 \frac{2}{\sqrt{x-1}}\,dx$

$= 2\lim_{t\to 1^+} 2\sqrt{x-1}\Big]_t^3$

$= 4\lim_{t\to 1^+}\left(\sqrt{2} - \sqrt{t-1}\right)$

$= 4\sqrt{2}$

so $\boxed{\int_1^3 \frac{2}{\sqrt{x-1}}\,dx = 4\sqrt{2}}$

(e) $I = \int_0^1 \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\,dx$

$\begin{cases} f = \ln x & g' = 1 \\ f' = \frac{1}{x} & g = x \end{cases}$

$I = \lim_{t\to 0^+}\left(x\ln x\Big]_t^1 - \int_t^1 dx\right)$

$= \lim_{t\to 0^+}\left(-t\ln t - 1 + t\right)$

$= -1 + 0 - \lim_{t\to 0^+}(t\ln t)$

$= -1 - \lim_{t\to 0^+}\frac{\ln t}{\frac{1}{t}}$ $\left(\frac{\infty}{\infty}\text{-case}\right)$

$= -1 - \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}}$

$= -1 + \lim_{t\to 0^+} t = -1$

so $\boxed{\int_0^1 \ln x\,dx = -1}$

(f) $\int_0^{\pi/2}\cos^2 y\,dy = \int_0^{\pi/2}\frac{1+\cos 2y}{2}\,dy$

$= \frac{1}{2}\left(\int_0^{\pi/2} dy + \int_0^{\pi/2}\cos 2y\,dy\right)$

$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{1}{2}\sin 2y\Big]_0^{\pi/2}\right)$

$= \frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4}$

$\boxed{\int_0^{\pi/2}\cos^2 y\,dy = \frac{\pi}{4}}$

(g) $I = \int \frac{x^3}{\sqrt{x^2+4}}\,dx$

let $\sqrt{x^2+4} = u$

$\frac{1}{2\sqrt{x^2+4}}\cdot 2x\,dx = du$

$\frac{x\,dx}{\sqrt{x^2+4}} = du$

then, $x^2+4 = u^2$

$x^2 = u^2-4$

$I = \int (u^2-4)\,du$

$= \frac{u^3}{3} - 4u + C$

$\boxed{I = \frac{1}{3}\sqrt{(x^2+4)^3} - 4\sqrt{x^2+4} + C}$

(h) $\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$LCD = x(x^2+1)^2$

$1 = A(x^2+1)^2 + x(Bx+C)(x^2+1) + x(Dx+E)$

$1 = Ax^4 + 2Ax^2 + A + (Bx^2+Cx)(x^2+1) + Dx^2 + xE$

$1 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + xE$

$1 = (A+B)x^4 + (2A+B+C+D)x^2 + Cx^3 + (C+E)x + A$

so $\begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ C+E=0 \\ A=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=0 \\ D=-1 \\ E=0 \end{cases}$

so $I = \int \frac{dx}{x(x^2+1)^2} =$

$= \int \frac{1}{x}\,dx + \int \frac{-x}{x^2+1}\,dx + \int \frac{-x}{(x^2+1)^2}\,dx$

$\left(\begin{array}{l} \text{let } x^2+1=u \\ 2x\,dx = du \end{array}\right)$

$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \frac{1}{2}\int u^{-2}\,du$

$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + C$

$\boxed{I = \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + C}$

(i) $I = \int_{\frac{5\pi}{6}}^{\pi} \frac{\cos^4 x}{\sqrt{1-\sin x}}\,dx$

$= \int_{5\pi/6}^{\pi} \frac{\cos^4 x\sqrt{1+\sin x}}{\sqrt{(1-\sin x)(1+\sin x)}}\,dx$

$= \int_{5\pi/6}^{\pi} \frac{\cos^4 x\sqrt{1+\sin x}}{|\cos x|}\,dx$ $\left(|\cos x| = -\cos x \text{ on } [\frac{5\pi}{6}, \pi]\right)$

$= -\int_{5\pi/6}^{\pi} \cos^3 x\sqrt{1+\sin x}\,dx$

$\begin{cases} \text{let } 1+\sin x = u \\ \cos x\,dx = du \\ \sin x = u-1 \\ \cos^2 x = 1-\sin^2 x \\ \quad = 1-(u-1)^2 = 2u-u^2 \\ x = \frac{5\pi}{6},\ u = \frac{3}{2} \\ x = \pi,\ u = 1 \end{cases}$

$I = -\int_{3/2}^{1} (2u-u^2)\sqrt{u}\,du$

$= \int_{1}^{3/2} (2u-u^2)u^{\frac{1}{2}}\,du$

$= 2\int_{1}^{3/2} (u^{3/2} - u^{5/2})\,du$

$= 2\left(\frac{2u^{\frac{5}{2}}}{5}\Big]_1^{3/2} - \frac{2u^{\frac{7}{2}}}{7}\Big]_1^{3/2}\right)$

$= \frac{4}{5}\left(\frac{3}{2}\right)^{5/2} - \frac{4}{5} - \frac{4}{7}\left(\frac{3}{2}\right)^{\frac{7}{2}} + \frac{4}{7}$

$\boxed{I = \frac{-8}{35} + \frac{4}{5}\left(\frac{3}{2}\right)^{5/2} - \frac{4}{7}\left(\frac{3}{2}\right)^{7/2}}$

(j) $I = \int \sec^3\theta \, d\theta$

$= \int \sec^2\theta \sec\theta \, d\theta$

$= \int (1+\tan^2\theta)\sec\theta \, d\theta$

$= \int \sec\theta \, d\theta + \int \sec\theta \tan^2\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| +$

$+ \int \tan\theta (\sec\theta \tan\theta) \, d\theta$

$\left[\begin{array}{ll} f = \tan\theta & g' = \sec\theta \tan\theta \\ f' = \sec^2\theta & g = \sec\theta \end{array}\right.$

$I = \ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta - I$

$2I = \ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta$

$$\boxed{I = \frac{1}{2}\ln|\sec\theta + \tan\theta| + \frac{1}{2}\sec\theta\tan\theta + C}$$

(k) $I = \int_{-\infty}^{0} x e^x \, dx$

$= \lim_{t\to-\infty} \int_t^0 x e^x \, dx$

$\left[\begin{array}{ll} f = x & g' = e^x \\ f' = 1 & g = e^x \end{array}\right.$

$= \lim_{t\to-\infty} \left(x e^x \Big|_t^0 - \int_t^0 e^x \, dx \right)$

$= \lim_{t\to-\infty} \left(0 - te^t - (1 - e^t) \right)$

$= \lim_{t\to-\infty} \left(e^t - 1 - te^t \right)$

$I = 0 - 1 - \lim_{t\to-\infty} te^t$ ($\infty \cdot 0$ case)

$= -1 - \lim_{t\to-\infty} \frac{t}{e^{-t}}$ ($\frac{\infty}{\infty}$ - case)

$= -1 - \lim_{t\to-\infty} \frac{1}{-e^{-t}} = -1 - \frac{1}{\infty} = -1$

so $\boxed{\int_{-\infty}^{0} x e^x \, dx = -1}$

(l) Know

$\sin(a+b) = \sin a \cos b + \sin b \cos a$

$\sin(a-b) = \sin a \cos b - \sin b \cos a$

(+) $\sin(a+b) + \sin(a-b) =$

$= 2 \sin a \cos b$

Therefore,

$\sin 2t \cos 4t = \frac{1}{2}\left(\sin 6t + \sin(-2t)\right)$

$= \frac{1}{2}\sin 6t - \frac{1}{2}\sin 2t$

$I = \int \sin 2t \cos 4t \, dt$

$= \frac{1}{2}\int \sin 6t \, dt - \frac{1}{2}\int \sin 2t \, dt$

$= \frac{1}{2}\cdot\frac{-\cos 6t}{6} - \frac{1}{2}\cdot\frac{-\cos 2t}{2} + C$

$$\boxed{I = \frac{1}{4}\cos 2t - \frac{1}{12}\cos 6t + C}$$

(m) $I=\int \frac{dX}{x^2+x+3}$

note that x^2+x+3 doesn't have real zeros.

$$x^2+x+3 = x^2+x+\frac{1}{4}+\frac{11}{4}$$
$$= \left(x+\frac{1}{2}\right)^2+\frac{11}{4}$$

$$I=\int\frac{dx}{(x+\frac{1}{2})^2+\frac{11}{4}}$$

Know $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$

Therefore,

$$I=\frac{2}{\sqrt{11}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{11}}{2}}+C$$

$$\boxed{I=\frac{2}{\sqrt{11}}\tan^{-1}\frac{2x+1}{\sqrt{11}}+C}$$

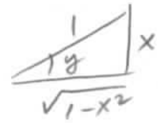
(n) $I=\int\sin^{-1}x\,dX$

let
$$\begin{cases} y=\sin^{-1}x, & y\in[-\frac{\pi}{2},\frac{\pi}{2}] \\ \sin y = x & x\in[-1,1] \\ \cos y\,dy = dX \end{cases}$$

$$I=\int y\cos y\,dy$$

let $\begin{cases} f=y & g'=\cos y \\ f'=1 & g=\sin y \end{cases}$

$$I=y\sin y-\int\sin y\,dy$$
$$= x\sin^{-1}x-\cos y+C$$
$$= x\sin^{-1}x-\cos(\sin^{-1}x)+C$$



$\cos y=\sqrt{1-x^2}$

so $\boxed{I=x\sin^{-1}x-\sqrt{1-x^2}+C}$

(10) (a) $\int_0^2\frac{dx}{1-x^2}\,dX=$

$$=\int_0^1\frac{dx}{1-x^2}+\int_1^2\frac{dX}{1-x^2}$$

let $I_1=\int_0^1\frac{dx}{1-x^2}$

$$=\lim_{t\to1^-}\int_0^t\frac{dx}{1-x^2}\,dx$$

$$\frac{1}{1-x^2}=\frac{A}{1-x}+\frac{B}{1+x}$$
$$1=A(1+x)+B(1-x)$$
$$1=A+Ax+B-Bx$$

$$\begin{cases} A-B=0 \\ A+B=1 \end{cases} \Rightarrow \begin{matrix} A=\frac{1}{2} \\ B=\frac{1}{2} \end{matrix}$$

$$I_1=\lim_{t\to1^-}\left(\int_0^t\frac{1}{2(1-x)}\,dx+\int_0^t\frac{1}{2(1+x)}\,dx\right)$$

$$=\lim_{t\to1^-}\left(-\frac{1}{2}\ln|1-x|\Big]_0^t+\frac{1}{2}\ln|1+x|\Big]_0^t\right)$$

$$=\lim_{t\to1^-}\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\Big]_0^t$$

$$=\lim_{t\to1^-}\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|=\infty$$

so I_1 = divergent $\Rightarrow$ $\boxed{I=\text{divergent}}$

(b) $\displaystyle\int_1^\infty \frac{dx}{x^5+2}$

$x^5+2 > x^5 \geq 1, \forall x \geq 1$

$\dfrac{1}{x^5+2} < \dfrac{1}{x^5}, \quad \forall x \geq 1$

and $\displaystyle\int_1^\infty \frac{1}{x^5}\,dx$ is convergent as $p>1$.

by Direct Comparison Test

$\Rightarrow \boxed{\displaystyle\int_1^\infty \frac{1}{x^5+2}\,dx \text{ is convergent}}$

(c) $\displaystyle\int_0^\infty \sin x\,dx = \lim_{t\to\infty}\int_0^t \sin x\,dx$

$= \displaystyle\lim_{t\to\infty}\left(-\cos x\Big]_0^t\right)$

$= \displaystyle\lim_{t\to\infty}\left(-\cos t + 1\right)$

but $\displaystyle\lim_{t\to\infty}\cos t$ does not exist

Therefore, $\boxed{\displaystyle\int_0^\infty \sin x\,dx \text{ is divergent}}$