Math 181 Spring 2012 Name:

QUIZ #2 @ 100 points

Write neatly. Show all work. No work, no credit given. Write all responses on separate paper. Clearly label the exercises. No graphing calculator allowed. No phone allowed.

1. The exercise gives parametric equations and parameter interval for the motion of a particle in the *xy*-plane.

   a) Identify the particle's path by finding a Cartesian equation for it.

   b) Graph the Cartesian equation. Indicate the portion of the graph traced by the particle and the direction of motion.

$$x = t + \frac{1}{t},\ y = t - \frac{1}{t},\ t > 0$$

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2. Find parametric equations and a parameter interval for the motion of a particle that starts at $(2,0)$ and traces the top half of the circle $x^2 + y^2 = 4$ four times.

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3. Find an equation for the line tangent to the curve at the point defined by the given value of t.

$$x = \frac{1}{t},\quad y = -2 + \ln t,\quad t = 1$$

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4. Find the Cartesian coordinates of the following point given in polar coordinates:

$$\left(-3, \frac{5\pi}{6}\right)$$

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5. a) Find the polar coordinates, $-\pi \le \theta < \pi$ and $r \ge 0$, of the following point given in Cartesian coordinates: $(5,-12)$

   b) Find the polar coordinates, $0 \le \theta < 2\pi$ and $r \le 0$, of the following point given in Cartesian coordinates: $\left(-1, \sqrt{3}\right)$

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6. Replace the polar equation with equivalent Cartesian equations. Then describe or identify the graph.

$$r^2 = 3r\cos\theta$$

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7. Replace the Cartesian equation with equivalent polar equations.

$$x^2 - y^2 = 1$$

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8. Graph the sets of points whose polar coordinates satisfy the inequalities:

   a) $0 \le \theta \le \frac{\pi}{6}, r \ge 0$

   b) $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ -1 \le r \le 1$

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9. a) Graph the curve $r = 1 - \cos\theta$. Show all work; in order to get credit, you must justify your polar coordinate graph by first showing either a Cartesian $r\theta$ -graph or a table of values. (You may want to check for symmetries.)

b) Find the slope of the curve at $\theta = \frac{\pi}{2}$ and sketch the tangent at that point on the graph.

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10. Graph the following curves. Show all work. ; in order to get credit, you must justify your polar coordinate graph by first showing either a Cartesian graph or a table of values . (You may want to check for symmetries.)

a) $r^2 = \sin 2\theta$ b) $r = \sin 2\theta$

c) $r = \frac{1}{2} + \cos\theta$ d) $r^2 = 4\cos 2\theta$

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11. a) First, graph the circle $r = 2\sin\theta$.

b) Then find the area of the region bounded by the circle for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$.

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12. a) Graph the circle $r = 6$ and the line $r = 3\csc\theta$. Show all work.

b) Find the common points of the two graphs.

c) Find the area inside the circle and above the line.

(181)

Quiz #2 - solutions

(1) $x = t + \frac{1}{t}, \; y = t - \frac{1}{t}, \; t > 0$

(a) $\left.\begin{aligned} x - y &= \frac{2}{t} \\ x + y &= 2t \end{aligned}\right\} \Rightarrow$

$(x-y)(x+y) = \frac{2}{t} \cdot 2t = 4$

so $\boxed{x^2 - y^2 = 4}$, $x = \sqrt{4+y^2}$

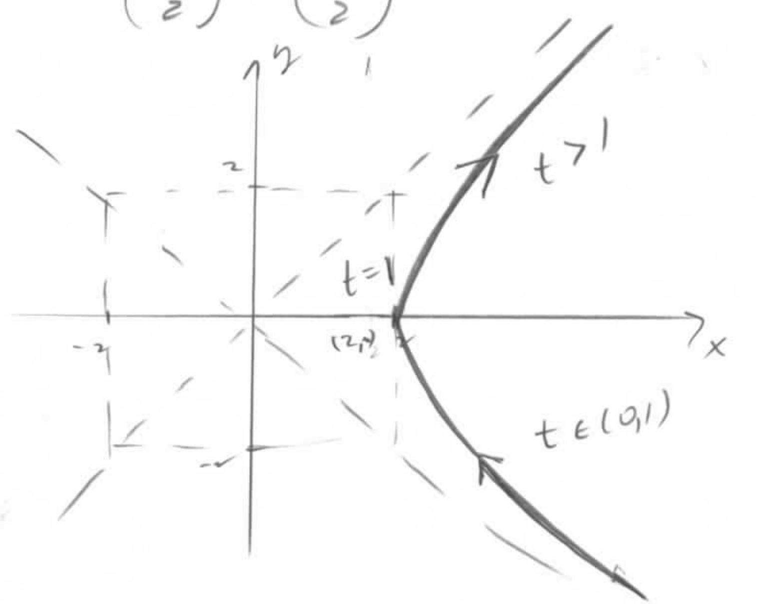
hyperbola;

Note that $x = t + \frac{1}{t} > 0$ for $\forall\, t > 0$

So just the right-branch of the hyperbola $\boxed{x = \sqrt{4+y^2}}$

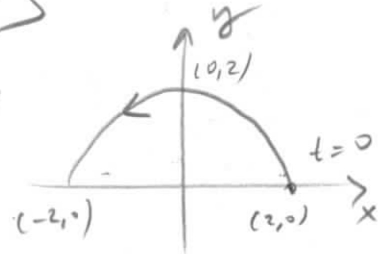
(b) $x^2 - y^2 = 4$

$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{2}\right)^2 = 1$



$x = \sqrt{4+y^2}$.

(2) $x^2 + y^2 = 4$



want $y \geq 0$

let $\begin{cases} x = 2\cos t \in [-2,2] \\ y = 2|\sin t| \in [0,2] \\ 0 \le t \le \pi \end{cases}$

(3) $x = \frac{1}{t}, \; y = -2 + \ln t, \; t = 1$

when $t = 1$, $x = 1$, $y = -2$

Need $m = \left.\frac{dy}{dx}\right|_{t=1}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t}}{-\frac{1}{t^2}} = -t$

$\left.\frac{dy}{dx}\right|_{t=1} = -1 = m$

Use $m = -1$, $(1,-2)$

$y - y_1 = m(x - x_1)$

$y + 2 = -(x-1)$

$\boxed{y = -x - 1}$ line tangent to the curve when $t=1$

(4) $(-3, \frac{5\pi}{6})$

know $r = -3$, $\theta = \frac{5\pi}{6}$

Therefore the point is in quadrant IV

want x, y

$x = r\cos\theta = -3\cos\frac{5\pi}{6}$

$= -3\left(-\cos\frac{\pi}{6}\right)$

$= -3\left(-\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$

$y = r\sin\theta = -3\sin\frac{5\pi}{6}$

$= -3\sin\frac{\pi}{6} = -3\cdot\frac{1}{2}$

The point is $\boxed{\left(\frac{3\sqrt{3}}{2}, \frac{-3}{2}\right)}$

(6) $(-1, \sqrt{3}) \in \text{II}$

know $x = -1$, $y = \sqrt{3}$

want $\theta \in [0, 2\pi)$, $r \leq 0$

$r^2 = x^2 + y^2 = 1 + 3 = 4$

$\left.\begin{array}{l} r = \pm 2 \\ \text{but } r \leq 0 \end{array}\right\} \Rightarrow r = -2$

$\tan\theta = \frac{y}{x} = -\sqrt{3}$

$\theta = -\frac{\pi}{3} + k\pi$

$r < 0$, so need to choose θ such that its ray is in quadrant IV

so choose $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

The point is $\boxed{\left(-2, \frac{5\pi}{3}\right)}$

(5) (a) $(5, -12) \in \text{IV}$

know $x = 5$, $y = -12$

want $\theta \in [-\pi, \pi)$ and $r \geq 0$

$r^2 = x^2 + y^2 = 25 + 144 = 169$

$\left.\begin{array}{l} r = \pm 13 \\ \text{but } r \geq 0 \end{array}\right\} \Rightarrow r = 13$

$\tan\theta = \frac{y}{x} = -\frac{12}{5}$

$\theta = \tan^{-1}\left(-\frac{12}{5}\right) = -\tan^{-1}\frac{12}{5} \approx -1.17$

$\theta \in [-\pi, \pi)$

so the point is

$\boxed{\left(13, -\tan^{-1}\frac{12}{5}\right)}$

(6) $r^2 = 3r\cos\theta$

$x^2 + y^2 = 3x$

$x^2 - 3x + y^2 = 0$

$x^2 - 3x + \frac{9}{4} + y^2 = \frac{9}{4}$

$\boxed{\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}}$

circle with center $\left(\frac{3}{2}, 0\right)$ and radius $\frac{3}{2}$

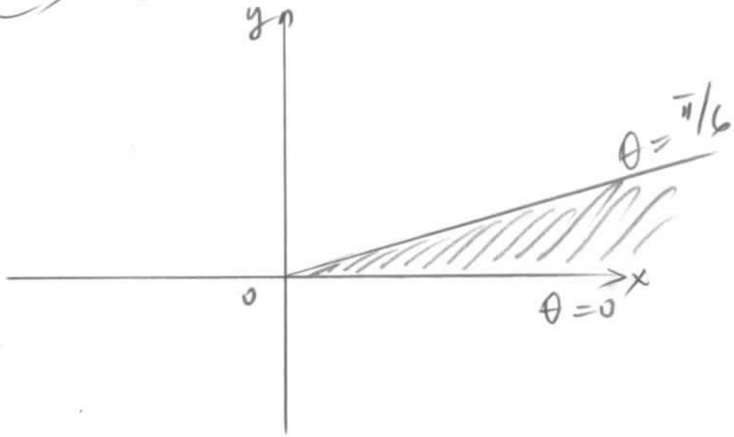
(7) $\left.\begin{array}{l} x^2-y^2=1 \\ x=r\cos\theta \\ y=r\sin\theta \end{array}\right\} \Rightarrow$

$r^2\cos^2\theta - r^2\sin^2\theta = 1$

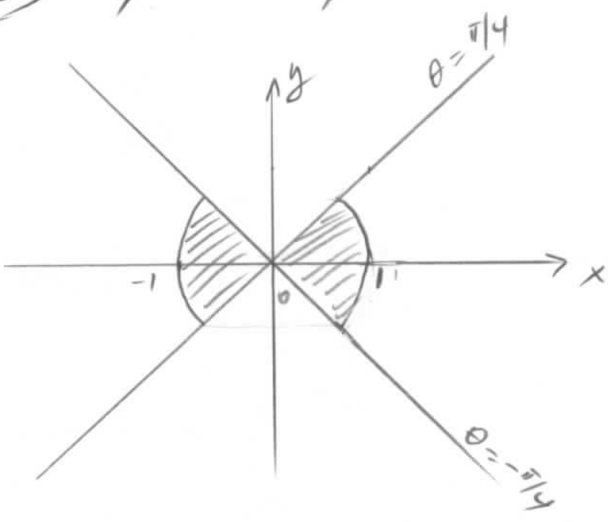
$r^2(\cos^2\theta - \sin^2\theta) = 1$

$\boxed{r^2\cos 2\theta = 1}$

(8) (a) $0 \le \theta \le \frac{\pi}{6}$, $r \geq 0$



(b) $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, $-1 \le r \le 1$



(9) (a) $r = 1-\cos\theta$

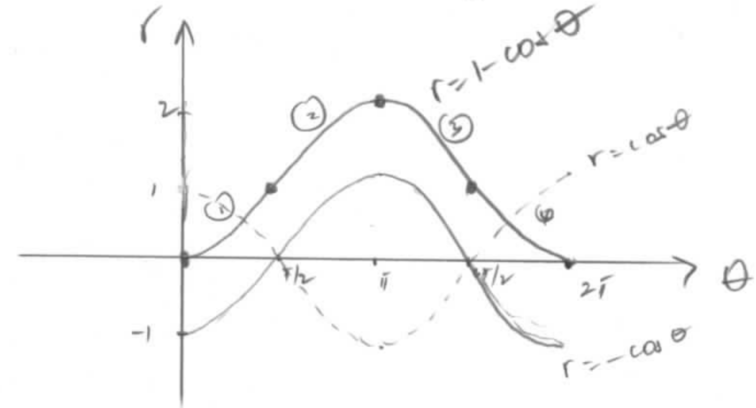
Note that $\theta \mapsto -\theta$,

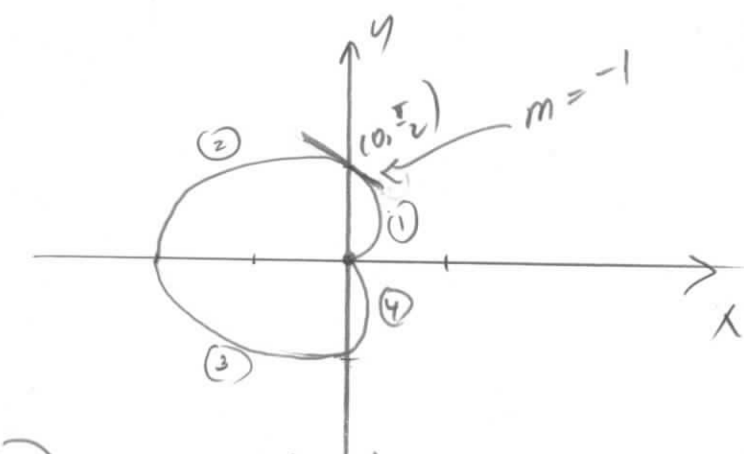
$r = 1-\cos(-\theta)$

$r = 1-\cos\theta$

so $(r,-\theta) \in$ graph

so the graph is symmetric about the x-axis





(b) want $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}}$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$\begin{cases} r = 1-\cos\theta \\ r\big|_{\theta=\frac{\pi}{2}} = 1 \end{cases}$

$\begin{cases} r' = \sin\theta \\ r'\big|_{\theta=\frac{\pi}{2}} = 1 \end{cases}$

$y = r\sin\theta$

$\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$

$\left.\frac{dy}{d\theta}\right|_{\theta=\frac{\pi}{2}} = 1\cdot\sin\frac{\pi}{2} + 1\cdot\cos\frac{\pi}{2} = 1$

$x = r\cos\theta$

$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta$

$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{2}} = 1\cos\frac{\pi}{2} - 1\cdot\sin\frac{\pi}{2} = -1$

Then, $\boxed{\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{1}{-1} = -1}$

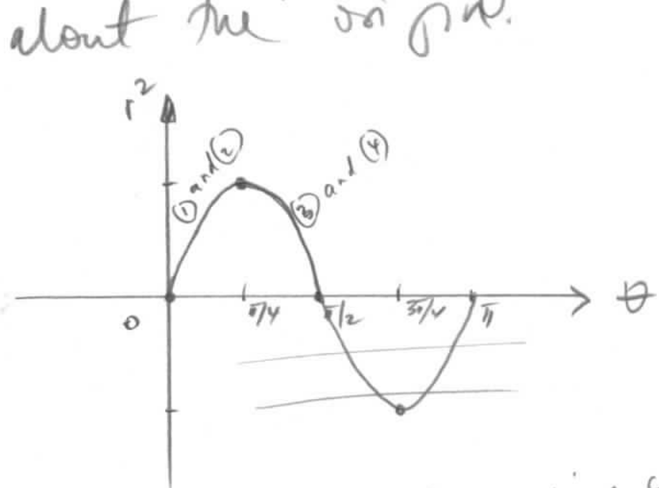
(10) (a) $r^2 = \sin 2\theta$

$r \mapsto -r$, $(-r)^2 = \sin 2\theta$

$r^2 = \sin 2\theta$

so $(-r, \theta) \in$ graph

so the graph is symmetric about the origin.



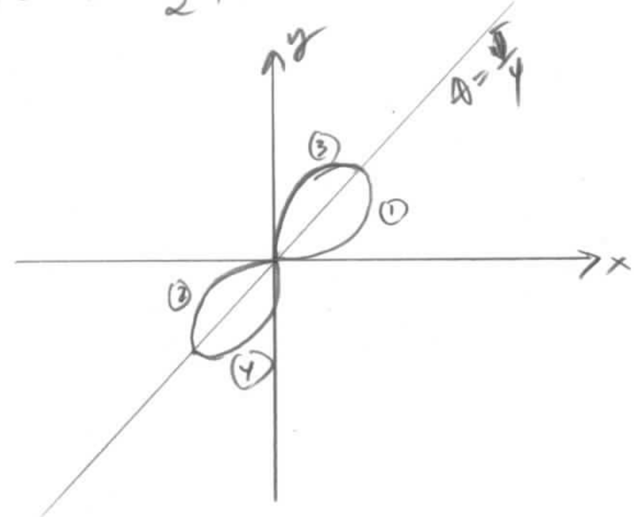
$r^2 = \sin 2\theta$ graph a sine curve of period π and amplitude 1

but $r^2 \geq 0$

when $\theta = 0$, $r^2 = 0$, $r = 0$

when $\theta = \frac{\pi}{4}$, $r^2 = 1$, $r = \pm 1$

when $\theta = \frac{\pi}{2}$, $r^2 = 0$, $r = 0$



(b) $r = \sin 2\theta$

$\left.\begin{array}{l} r \mapsto -r \\ \theta \mapsto \pi - \theta \end{array}\right\} \Rightarrow -r = \sin 2(\pi - \theta)$

$-r = \sin(2\pi - 2\theta)$

$-r = \sin(-2\theta)$

$-r = -\sin 2\theta$

$r = \sin 2\theta$

so $(-r, \pi - \theta) \in$ graph

so the graph is symmetric about the x-axis

$\left.\begin{array}{l} r \mapsto -r \\ \theta \mapsto -\theta \end{array}\right\} \Rightarrow -r = \sin(-2\theta)$

$-r = -\sin 2\theta$

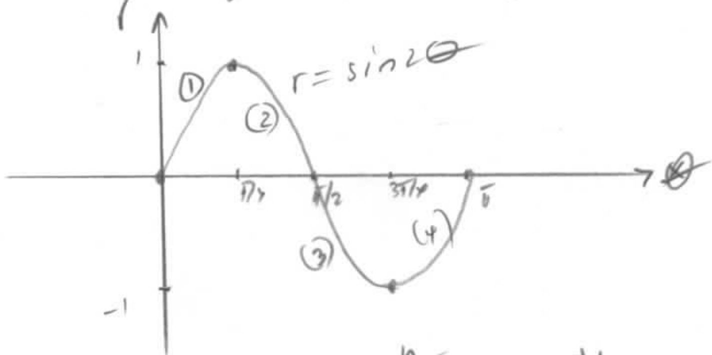
$r = \sin 2\theta$

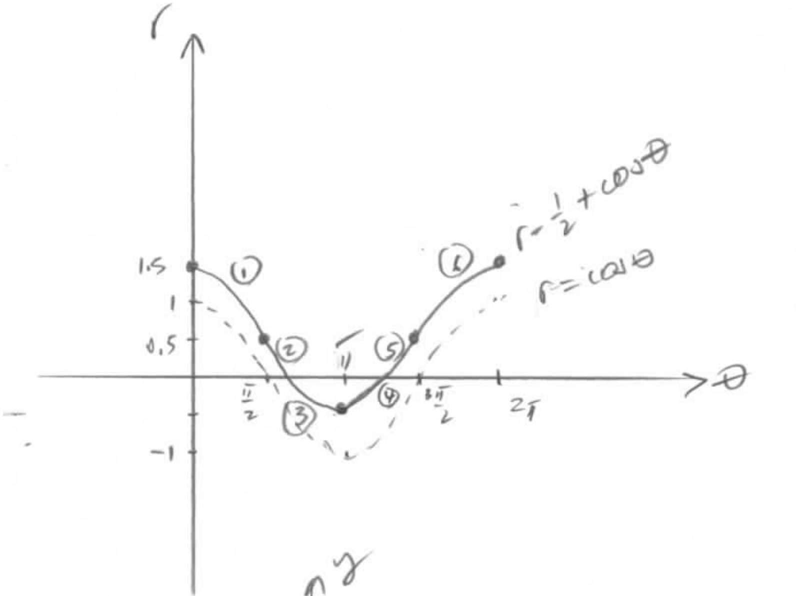
so $(-r, -\theta) \in$ graph

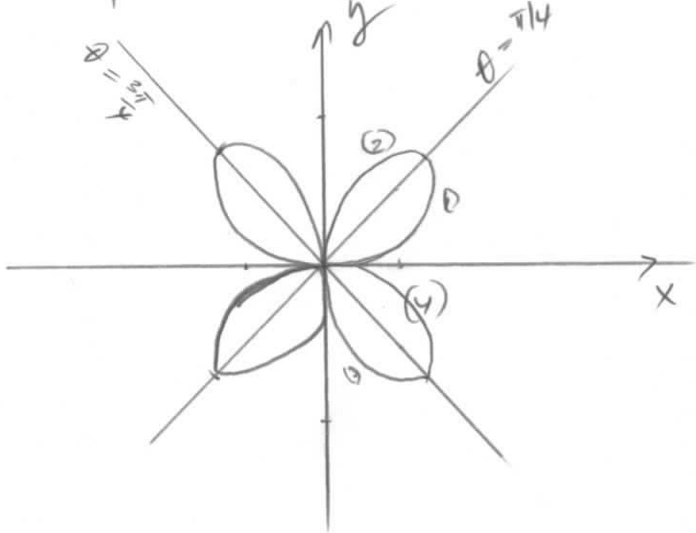
so the graph is sym. about the y axis

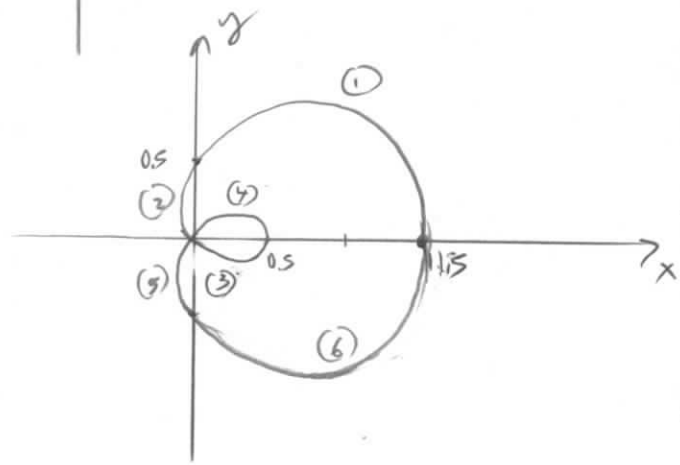
$r = \sin 2\theta$

graph a sine curve of period π and amplitude 1









ⓒ $r = \frac{1}{2} + \cos\theta$

$\theta \mapsto -\theta$, $r = \frac{1}{2} + \cos(-\theta)$

$r = \frac{1}{2} + \cos\theta$

so $(r, -\theta) \in$ graph

so the graph is symmetric about x axis

Graph a cosine curve of period 2π and amplitude 1, shifted up $\frac{1}{2}$

ⓓ $r^2 = 4\cos 2\theta$

$r \mapsto -r$, then $(-r, \theta) \in$ graph

so the graph is symmetric about the origin

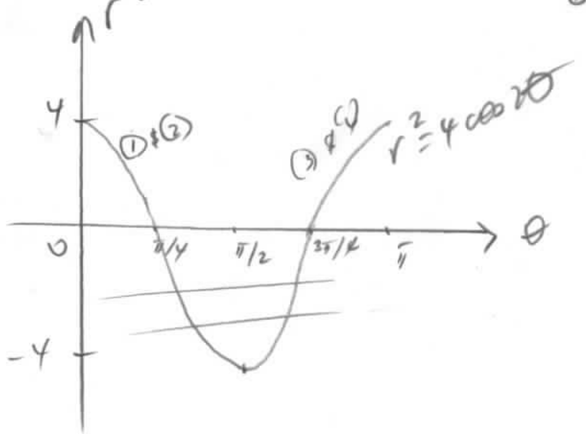
$r \mapsto -r$, $\theta \mapsto -\theta$, $(-r, -\theta) \in$ graph

so the graph is symmetric about the y-axis

$\theta \mapsto -\theta$, $(r, -\theta) \in$ graph

so the graph is symmetric about x-axis.

. we'll graph a cosine curve of amplitude ~~4~~ and period π



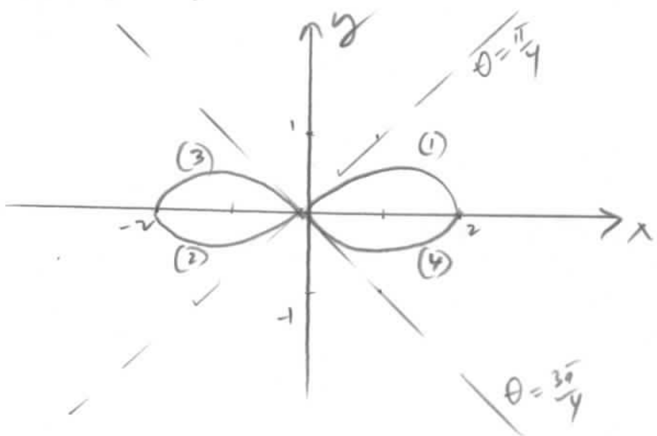
$r^2 \geqslant 0$

when $\theta = 0$, $r^2 = 4$, $r = \pm 2$

when $\theta = \frac{\pi}{4}$, $r^2 = 0$, $r = 0$

when $\theta = \frac{3\pi}{4}$, $r^2 = 0$, $r = 0$

when $\theta = \pi$, $r^2 = 4$, $r = \pm 2$



(11) (a) $r = 2\sin\theta$

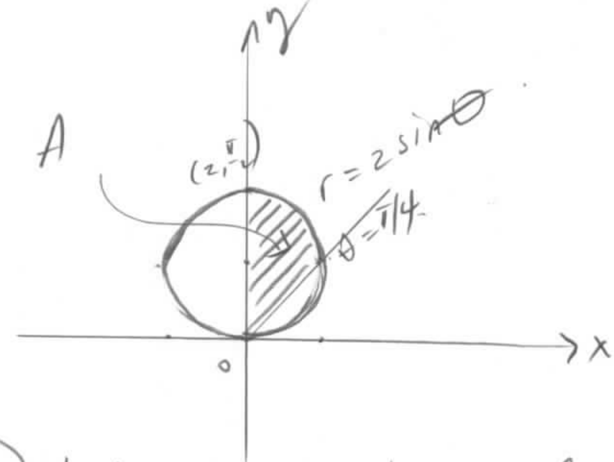
then $r^2 = 2r\sin\theta$

$x^2 + y^2 - 2y = 0$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$

circle with center $(0,1)$ and radius 1



(b) let A = area bounded by the circle for $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

$$A = \int_{\pi/4}^{\pi/2} \frac{1}{2} (2\sin\theta)^2 \, d\theta$$

$$= 2\int_{\pi/4}^{\pi/4} \sin^2\theta \, d\theta$$

$$= 2\int_{\pi/4}^{\pi/2} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (1-\cos 2\theta) \, d\theta$$

$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= \left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\pi}{4} + \frac{1}{2}$$

so $\boxed{A = \frac{\pi}{4} + \frac{1}{2} \text{ sq. units}}$

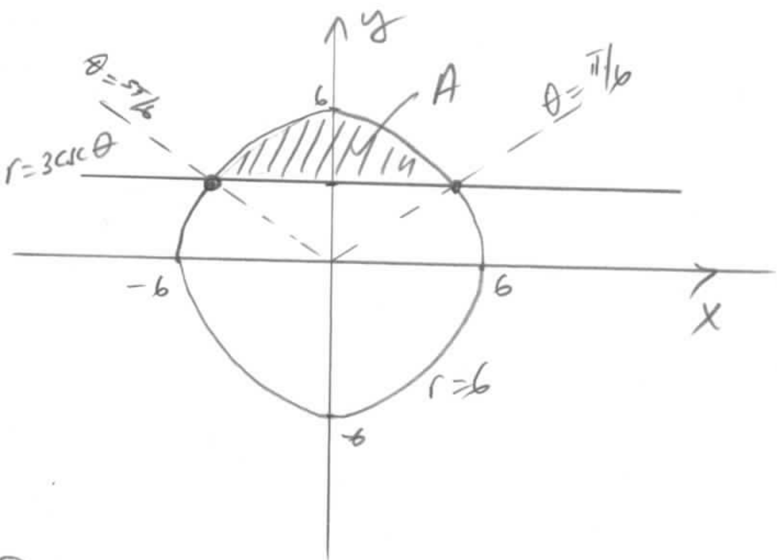
(12) (a) $r=6$ circle of center O and radius 6

$r = 3\csc\theta$

$r = \dfrac{3}{\sin\theta}$

$r\sin\theta = 3$

$y = 3$ horizontal line



(b) $3\csc\theta = 6$

$\sin\theta = \dfrac{1}{2}$

$\boxed{\theta = \pi/6 \text{ or } \dfrac{5\pi}{6}}$

(c) let A = area

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(6^2 - (3\csc\theta)^2\right) d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} \frac{1}{2}\left(36 - 9\csc^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(36 - 9\csc^2\theta\right) d\theta$$

$$= \Big[36\theta - 9(-\cot\theta)\Big]_{\pi/6}^{\pi/2}$$

$$= 36\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + 9\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{6}\right)$$

$$= 12\pi - 9\cdot\sqrt{3}$$

$\therefore$ $\boxed{A = 12\pi - 9\sqrt{3} \text{ sq. units}}$