TEST #1 @ 150 points

Write neatly. Show all work. **Write all proofs on separate paper. Label each exercise.**

1. a) Write the definition of

$$\lim_{x \to c} f(x) = L.$$

b) Write the mathematical definition of:

The function $y = f(x)$ *is continuous at* $x = a$.

c) When is a function $y = f(x)$ discontinuous at $x = a$?

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2. a) Prove that $\lim_{x \to 0} x^6 \sin \frac{2}{x} = 0$.

b) Prove that $\lim_{t \to \infty} \frac{\cos t}{t} = 0$

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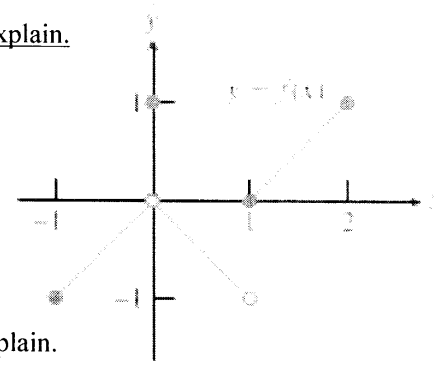
3. Use the following graph to answer the questions. <u>If a limit does not exist, explain.</u>



a) $\lim_{x \to 0} f(x)$

b) $\lim_{x \to 1} f(x)$

c) Is the function f continuous at $x = 0$? Explain using the definition.

d) At what points in the interval $(-1, 2)$ is the function discontinuous? Explain.

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4. Let $f(x) = \begin{cases} 2x + 1 & \text{if } x \neq -2 \\ k & \text{if } x = -2 \end{cases}$. Answer the following:

Find k such that f is continuous at $x = -2$. Do not just write an answer, show a correct mathematical proof.

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5. Find an equation for the tangent line to the curve $g(x) = \frac{x}{x-2}$ at $(3, 3)$.

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6. Complete the following . Do not prove. If a limit does not exist, say so, and explain why.

a) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} =$

c) $\lim\limits_{y \to 0^+} \dfrac{1}{y} =$

e) $\lim\limits_{x \to \infty} e^x =$

b) $\lim\limits_{x \to -\infty} \dfrac{1}{x} =$

d) $\lim\limits_{x \to 0^-} \dfrac{1}{x} =$

f) $\lim\limits_{x \to \infty} \cos x =$

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7. Find the following limits. Do not just write an answer, show proof.

a) $\lim\limits_{x \to 4^+} \dfrac{(x+3)(4-x)}{|x-4|}$

b) $\lim\limits_{\theta \to 0} \dfrac{2\theta + 4\theta\cos\theta}{\sin\theta\cos\theta}$

c) $\lim\limits_{x \to \pi} \dfrac{\cos x}{1-\pi}$

d) $\lim\limits_{a \to -2} \sqrt{4a^2 - 3}$

e) $\lim\limits_{x \to 0} \dfrac{\tan 3x}{\sin 7x}$

f) $\lim\limits_{x \to 0} \dfrac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

g) $\lim\limits_{x \to 1} \left(\sqrt{x^2 + x} - \sqrt{x^2 - x} \right)$

h) $\lim\limits_{x \to \infty} \left(x - \sqrt{x^2 - 1} \right)$

i) $\lim\limits_{t \to -\infty} \dfrac{t+2}{\sqrt{9t^2 + 1}}$

j) $\lim\limits_{a \to \infty} e^{-3a}$

k) $\lim\limits_{x \to 1} \sin^{-1}\left(\dfrac{1-\sqrt{x}}{1-x} \right)$

l) $\lim\limits_{c \to 1^+} \left(\dfrac{1}{c-1} - \dfrac{1}{c^3 - 1} \right)$

m) $\lim\limits_{x \to \infty} \dfrac{2x^3 - x^2 + x + 7}{3x^3 + 50x^2 - 100}$

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8. Does the graph of $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ have a vertical tangent at $(0,1)$? Do not just write an answer. Prove it.

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9. Let $g(x) = x^2 + 4x - 1$. At what points does the graph of the function have a horizontal tangent?

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TEST 1 - SOLUTIONS

① (a) $\lim_{x\to c} f(x) = L$ iff

the values of $f(x)$ get arbitrarily close to L by taking x to be sufficiently close to c (on either side of c), but not equal to c.

(b) $y = f(x)$

f is continuous at $x = a$ iff $\lim_{x\to a} f(x) = f(a)$

(c) f is discontinuous at $x = a$ if

- $f(a)$ is not defined

OR

- $\lim_{x\to a} f(x)$ does not exist

OR

- $\lim_{x\to a} f(x) \neq f(a)$

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② (a) $\lim_{x\to 0} x^6 \sin\frac{2}{x} = 0$

Proof

Note that the Product Rule cannot be applied here as $\lim_{x\to 0}\frac{2}{x}$ does not exist.

We'll use the Squeeze Theorem:

We know

$-1 \le \sin\frac{2}{x} \le 1$ for any $x \neq 0$

Multiply each side by x^6 (note $x^6 > 0$ as $x \to 0$)

$-x^6 \le x^6 \sin\frac{2}{x} \le x^6$

$\lim_{x\to 0}(-x^6) = \lim_{x\to 0} x^6 = 0$

Therefore, by the Squeeze Th. (Sandwich Th.),

$\lim_{x\to 0} x^6 \sin\frac{2}{x} = 0$

(b) $\lim_{t\to\infty}\frac{\cos t}{t} = 0$

Proof

Note that the Quotient Rule cannot be applied as $\lim_{t\to\infty}\cos t$ does not exist.

We'll use the Squeeze Theorem.

We know $-1 \le \cos t \le 1$ for any t

Multiply each side by $\frac{1}{t}$ (note that $\frac{1}{t} > 0$ as $t \to \infty$)

$-\frac{1}{t} \le \frac{\cos t}{t} \le \frac{1}{t}$

$$\lim_{t\to\infty} \frac{-1}{t} = \lim_{t\to\infty} \frac{1}{t} = \frac{1}{\infty} = 0$$

Therefore, by the Squeeze Theorem $\Rightarrow$

$$\lim_{t\to\infty} \frac{\cos t}{t} = 0$$

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(3) (a) $\lim_{x\to 0} f(x) = 0$

(b) $\lim_{x\to 1} f(x)$ does not exist

because $\lim_{x\to 1^-} f(x) = -1$ and

$\lim_{x\to 1^+} f(x) = 0$

(c) f not continuous at $x=0$

because $\lim_{x\to 0} f(x) \neq f(0)$

$\lim_{x\to 0} f(x) = 0$ and

$f(0) = 1$

(d) f is discontinuous

at $x=0$ and $x=1$

At $x=0$ because $\lim_{x\to 0} f(x) \neq f(0)$

At $x=1$ because

$\lim_{x\to 1} f(x)$ does not exist

(4) $f(x) = \begin{cases} 2x+1, & x \neq -2 \\ k, & x = -2 \end{cases}$

Solution

f - continuous at $x=-2$

iff $\lim_{x\to -2} f(x) = f(-2)$

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} (2x+1)$$

$$= 2(-2)+1$$

$$= -3$$

$f(-2) = k$

Therefore, $k = -3$

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(5) $g(x) = \frac{x}{x-2}$

Tangent line at $(3,3)$

Solution

For the equation of the line we need: a point $(3,3)$

the slope m

$$m = \lim_{x\to 3} \frac{g(x)-g(3)}{x-3}$$

$$= \lim_{x\to 3} \frac{\frac{x}{x-2} - 3}{x-3}$$

$$= \lim_{x\to 3} \frac{\frac{x-3(x-2)}{x-2}}{x-3}$$

$$= \lim_{x\to 3} \frac{x-3x+6}{(x-2)(x-3)}$$

$$= \lim_{x\to 3} \frac{6-2x}{(x-2)(x-3)}$$

$$= \lim_{x\to 3} \frac{2\cancel{(3-x)}(-1)}{(x-2)\cancel{(x-3)}}$$

$$= \lim_{x\to 3} \frac{-2}{x-2} = \frac{-2}{3-2} = -2$$

so $m = -2$

We'll use $(3,3)$ and $m=-2$

$y - y_1 = m(x-x_1)$

$y-3 = -2(x-3)$

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(6) (a) $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

(b) $\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$

(c) $\lim_{y\to 0^+} \frac{1}{y} = \frac{1}{0^+} = \infty$

(d) $\lim_{x\to 0^-} \frac{1}{x} = \frac{1}{0^-} = -\infty$

(e) $\lim_{x\to\infty} e^x = e^{\infty} = \infty$

(f) $\lim_{x\to\infty} \cos x$ - does not exist
as $\cos x$ oscillates between
1 and -1

(7) (a)

$$\lim_{x\to 4^+} \frac{(x+3)(4-x)}{|x-4|} =$$

$$\left(\begin{array}{l} x \to 4^+ \\ x > 4 \\ x-4>0, \text{ so } |x-4| = x-4 \end{array} \right)$$

$$= \lim_{x\to 4^+} \frac{(x+3)\cancel{(4-x)}(-1)}{\cancel{x-4}}$$

$$= \lim_{x\to 4^+} (-1)(x+3)$$

$$= (-1)(4+3) = -7$$

(b)
$$\lim_{\theta\to 0} \frac{2\theta + 4\theta\cos\theta}{\sin\theta\cos\theta} =$$

$$= \lim_{\theta\to 0} \frac{2\theta}{\sin\theta\cos\theta} + \lim_{\theta\to 0} \frac{4\theta\cancel{\cos\theta}}{\sin\theta\cancel{\cos\theta}}$$

$$= \frac{2}{1} \cdot \lim_{\theta\to 0} \frac{\theta}{\sin\theta} + 4 \lim_{\theta\to 0} \frac{\theta}{\sin\theta}$$

$$= 2 \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} + 4 \cdot \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}}$$

$$= 2\cdot 1 + 4\cdot 1$$

$$= 6$$

(c) $\lim_{x \to \pi} \frac{\cos x}{1-\pi} = \frac{\cos \pi}{1-\pi} = \frac{-1}{1-\pi}$

(d) $\lim_{a \to -2} \sqrt{4a^2-3} = \sqrt{4(-2)^2-3}$

$= \sqrt{13}$

(e) $\lim_{x \to 0} \frac{\tan 3x}{\sin 7x} =$

$= \lim_{x \to 0} \frac{\frac{\sin 3x}{\cos 3x}}{\sin 7x}$

$= \lim_{x \to 0} \frac{\sin 3x}{\cos 3x \sin 7x}$

$= \frac{1}{1} \cdot \lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$

$= \lim_{x \to 0} \frac{\frac{\sin 3x}{3x} \cdot 3\cancel{x}}{\frac{\sin 7x}{7x} \cdot 7\cancel{x}}$

$= \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 7x}{7x}} \cdot \lim_{x \to 0} \frac{3}{7}$

$= \frac{1}{1} \cdot \frac{3}{7} = \frac{3}{7}$

(f) $\lim_{x \to 0} \frac{x^2-4x+4}{x^3+5x^2-14x} = \left(\frac{4}{0}\right)$

$= \lim_{x \to 0} \frac{(x-2)^2}{x(x^2+5x-14)}$

$= \lim_{x \to 0} \frac{(x-2)^{\cancel{2}}}{x(x+7)\cancel{(x-2)}}$

$= \frac{-2}{0(7)} = \frac{-2}{0}$ does not exist

$\lim_{x \to 0^+} \frac{x^2-4x+4}{x^3+5x^2-14x} = \frac{-2}{0^+} = -\infty$

and

$\lim_{x \to 0^-} \frac{x^2-4x+4}{x^3+5x^2-14x} = \frac{-2}{0^-} = +\infty$

(g) $\lim_{x \to 1} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) =$

$= \sqrt{1+1} - \sqrt{1-1} = \sqrt{2}$

(h) $\lim_{x \to \infty} \left(x - \sqrt{x^2-1}\right) = \infty - \infty$ (special case)

$= \lim_{x \to \infty} \left(x - \sqrt{x^2-1}\right) \cdot \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$

$= \lim_{x \to \infty} \frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}}$

$= \lim_{x \to \infty} \frac{+1}{x+\sqrt{x^2-1}} = \frac{+1}{\infty} = 0$

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(i) $\lim_{t\to-\infty} \frac{t+2}{\sqrt{9t^2+1}}$ $\left(= \frac{\infty}{\infty} \text{ special case}\right)$

$= \lim_{t\to-\infty} \frac{t+2}{\sqrt{t^2(9+\frac{1}{t^2})}}$

$= \lim_{t\to-\infty} \frac{t(1+\frac{2}{t})}{|t|\sqrt{9+\frac{1}{t^2}}}$

(when $t\to-\infty$, $|t| = -t$)

$= \lim_{t\to-\infty} \frac{\cancel{t}(1+\frac{2}{t})}{-\cancel{t}\sqrt{9+\frac{1}{t^2}}}$

$= \lim_{t\to-\infty} \frac{1+\frac{2}{t}}{-\sqrt{9+\frac{1}{t^2}}} = \frac{1+0}{-\sqrt{9+0}}$

$= -\frac{1}{3}$

(j) $\lim_{a\to\infty} e^{-3a} = e^{-\infty} = 0$

(k) $\lim_{x\to1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) =$

$= \sin^{-1}\left(\lim_{x\to1} \frac{1-\sqrt{x}}{1-x}\right)$ $\frac{0}{0}$ special case

$= \sin^{-1}\left(\lim_{x\to1} \frac{\cancel{1-\sqrt{x}}}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}\right)$

$= \sin^{-1}\left(\lim_{x\to1} \frac{1}{1+\sqrt{x}}\right)$

$= \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

(l) $\lim_{c\to1^+} \left(\frac{1}{c-1} - \frac{1}{c^3-1}\right) =$

$\infty - \infty$ special case

$= \lim_{c\to1^+} \left(\frac{1}{c-1} - \frac{1}{(c-1)(c^2+c+1)}\right)$

$= \lim_{c\to1^+} \frac{c^2+c+\cancel{1}-\cancel{1}}{(c-1)(c^2+c+1)}$

$= \lim_{c\to1^+} \frac{c(c+1)}{(c-1)(c^2+c+1)} = \frac{1\cdot2}{0^+(3)}$

$= \infty$

(m) $\lim_{x\to\infty} \frac{2x^3-x^2+x+7}{3x^3+50x^2-100} =$

($\frac{\infty}{\infty}$ - special case)

$= \lim_{x\to\infty} \frac{\cancel{x^3}(2-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3})}{\cancel{x^3}(3+\frac{50}{x}-\frac{100}{x^3})}$

$= \lim_{x\to\infty} \frac{2-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}{3+\frac{50}{x}-\frac{100}{x^3}}$

$= \frac{2}{3}$

(8) $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

Solution

The graph has a vertical tangent iff the slope of the tangent at (0,0) is ∞ or $-\infty$

Look at

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{1-1}{x}$$

$$= \lim_{x \to 0^+} \frac{0}{x} = \lim_{x \to 0^+} 0 = 0$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{0-1}{x}$$

$$= \frac{-1}{0^-} = \infty$$

Therefore, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist;

the graph doesn't have <u>any</u> kind of tangent at (0,0).

(9) $g(x) = x^2 + 4x - 1$

Solution

The graph has a horizontal tangent at $(a, g(a))$ iff the slope of the tangent is 0

$$m = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h \to 0} \frac{((a+h)^2 + 4(a+h) - 1) - (a^2 + 4a - 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} + 2ah + h^2 + \cancel{4a} + 4h - \cancel{1} - \cancel{a^2} - \cancel{4a} + \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 + 4h}{h}$$

$$= \lim_{h \to 0} (2a + h + 4)$$

$$= 2a + 4$$

So we want

$$2a + 4 = 0$$

$$a = -2$$

So the graph has a horizontal tangent when $x = -2$, $y = -5$, at $(-2, -5)$.