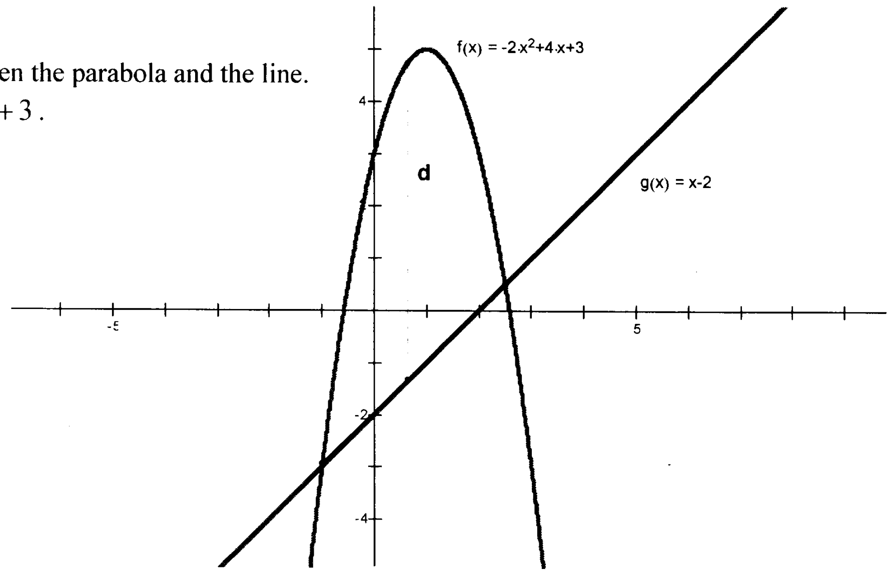
TEST #1 @ 175 points

Write neatly. Show all work. **Write all responses on separate paper.** **Clearly label the exercises.**

1. Find the maximum vertical distance d between the parabola and the line.
The equation of the parabola is $y = -2x^2 + 4x + 3$.
The equation of the line is $y = x - 2$.



2. Suppose $f(x) = \dfrac{x-1}{x-2}$, $g(x) = \sqrt{x+1}$. Answer the following:
   a) What is the domain of each function?
   b) Find $(f \circ g)(x)$ and its domain.
   c) Find $f^{-1}(x)$ and $g^{-1}(x)$.
   d) Prove algebraically that the function g is one-to-one.

3. Assume a function f is odd and another function g is even. Find whether fg and $\dfrac{f}{g}$ are even, odd, or neither. Show complete proof for each function separately.

4. Let $f(x) = x^7 - 4x^6 - 3x^5 + 10x^4 + 8x^3$. Be a polynomial function. Do the following:
   a) Graph the polynomial. Show all work: domain, end behavior, intercepts, behavior near the intercepts, test points (if any)). Organize all the information in a table of values.
   b) Solve the following inequalities: $f(x) \geq 0, f(x) < 0$

5. Solve the following equations:
   a) $9^{x^2} = 3^{3x+2}$
   b) $x^3\left(4e^{4x}\right) + 3x^2e^{4x} = 0$
   c) $\ln x = 1 - \ln(x+2)$
   d) $\log x - \log(x+1) = 3\log 4$
   e) $A = Ba^{Ct} + D$ solve for t.

6. Let $f(x) = \dfrac{x^2 - 3x - 4}{x^2 + x - 6}$ be a rational function. Do the following:

a) Graph the function $f(x) = \dfrac{x^2 - 3x - 4}{x^2 + x - 6}$. Show all work: domain, asymptotes, intercepts, test points (if any).

b) Solve the following inequalities: $f(x) \geq 0, f(x) < 0$

7. Let $f(x) = 2^{x+1} - 1$. Do the following:

a) Graph the function. You could plot points (clearly label them) or use transformations (clearly explain).
b) State the domain, range, and asymptote.
c) Find the exact *x*- and *y*-intercepts (if any).
d) Does the function have an inverse? Explain. Find the inverse function $f^{-1}(x)$.
e) Graph the inverse function showing how it can be obtained from the graph of *f*.
f) Find the exact *x*- and *y*-intercepts for $f^{-1}(x)$ (if any).

8. The growth in height of trees is frequently described by a logistic equation. Suppose the height *h* (in feet) of a tree at age *t* (in years) is

$$h = \frac{120}{1 + 200e^{-0.2t}}.$$

a) What is the height of the tree at age 10?
b) At what age is the height 50 feet?

9. The cost of a 30-second television advertisement during the Super Bowl is given for a few years: 550 thousand dollars in 1986, 1085 thousand dollars in 1996, 2100 thousand dollars in 2001, and 2400 thousand dollars in 2005.
Answer the following questions:

a) Is the cost of a 30-second TV advertisement (in thousand dollars) a function of the year? Explain.
b) Draw a table of values for the given data .
c) Does the cost depend linearly on the year? Why or why not? How can you be sure?
d) Determine a line that approximates the data - that is, find its equation. You may choose two of the given data.
e) Predict the cost of a 30-second commercial in 2009.
f) Does the function have an inverse? Explain.

extra credit
g) Find the inverse function (use your equation from part (d) above.). What is the meaning of the inverse function?
h) Find $f^{-1}(2200)$ and explain what it means.

10. The table shows the cost of a taxi ride as a function of miles traveled. Answer the following:

m	0	1	2	3	4	5
C(m)	0	2.50	4	5.50	7	8.50

a) What does $C(3.5)$ mean in practical terms? Estimate $C(3.5)$.

extra credit
b) What does $C^{-1}(3.5)$ mean? Estimate $C^{-1}(3.5)$.

11. Find a way to draw a rectangle in a rectangular coordinate system such that the figure is:

extra credit
a) symmetric about the y-axis only.
b) symmetric about the x-axis only.
c) symmetric about both axes and the origin.
d) symmetric about the origin only.

MATH 160

TEST 1 – SOLUTIONS

① let $y_1 = -2x^2+4x+3$

$y_2 = x-2$

then $d = y_1 - y_2$

$d = (-2x^2+4x+3)-(x-2)$

$d = -2x^2+3x+5$

- quadratic equation in x
- its graph is a parabola

opening down, so the maximum occurs at the vertex $V(x_v, d_v)$

$x_v = \frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{3}{4}$

$d_{max} = d_v = -2\left(\frac{3}{4}\right)^2 + 3\frac{3}{4} + 5$

$= -\frac{9}{8} + \frac{9}{4} + 5 = \frac{-9+18+40}{8}$

$\boxed{d_{max} = \frac{49}{8} = 6.125}$

② $f(x) = \frac{x-1}{x-2}$, $g(x) = \sqrt{x+1}$

a) Domain of f:

condition: $x-2 \neq 0$

$x \neq 2$

$\boxed{x \in \mathbb{R} \setminus \{2\}}$

Domain of g

Condition: $x+1 \geq 0$, $x \geq -1$

$\boxed{x \in [-1, \infty)}$

b) $(f \circ g)(x) = f(g(x))$

$= f(\sqrt{x+1})$

$\boxed{(f \circ g)(x) = \frac{\sqrt{x+1}-1}{\sqrt{x+1}-2}}$

Domain: $\begin{cases} x \in \text{Domain of } g \\ \text{and} \\ g(x) \in \text{Domain of } f \end{cases}$

$\begin{cases} x \in [-1,\infty) \\ \text{and} \\ \sqrt{x+1} \neq 2 \end{cases} \iff \begin{cases} x \in [-1,\infty) \\ \text{and} \\ x \neq 3 \end{cases}$

$\left(\sqrt{x+1} = 2\right)^2$

$x+1 = 4$

$x = 3$

therefore, for $f \circ g$ the domain is

$[-1,\infty) \setminus \{3\}$ OR

$\boxed{[-1,3) \cup (3,\infty)}$

c) $g(x) = \sqrt{x+1}$, $x \geq -1$

1st let $y = \sqrt{x+1}$ and solve for x

$y^2 = x+1$

$x = y^2 - 1$

3rd $x \leftrightarrow y$

$y = x^2 - 1$ $\boxed{g^{-1}(x) = x^2 - 1}$

$f(x) = \frac{x-1}{x-2}$

1st let $y = \frac{x-1}{x-2}$

2nd solve for x:

$y(x-2) = x-1$

$yx - 2y = x - 1$

$yx - x = 2y - 1$

$x(y-1) = 2y-1$

$x = \frac{2y-1}{y-1}$

3rd $x \Leftrightarrow y$

$y = \frac{2x-1}{x-1}$ $\boxed{f^{-1}(x) = \frac{2x-1}{x-1}}$

d) $g(x) = \sqrt{x+1}$, $x \geq -1$

is one-to-one

Proof

let x_1 and $x_2 \geq -1$ such that $g(x_1) = g(x_2)$

$\Rightarrow \sqrt{x_1+1} = \sqrt{x_2+1}$

$\Rightarrow x_1+1 = x_2+1$

$\Rightarrow x_1 = x_2$

therefore $g(x) = \sqrt{x+1}$ is a one-to-one function

③ f = odd, g = even

fg, f/g

Proof

f = odd $\Rightarrow f(-x) = -f(x)$ for any x

g = even $\Rightarrow g(-x) = g(x)$ for any x

$(fg)(-x) = f(-x)\,g(-x)$

$= -f(x)\,g(x)$

$= -(fg)(x)$

therefore fg is odd

$\left(\frac{f}{g}\right)(-x) = \frac{f(-x)}{g(-x)}$

$= \frac{-f(x)}{g(x)}$

$= -\left(\frac{f}{g}\right)(x)$

therefore $\frac{f}{g}$ is odd.

(4) $f(x) = x^7 - 4x^6 - 3x^5 + 10x^4 + 8x^3$

$f(x) = x^3(\underbrace{x^4 - 4x^3 - 3x^2 + 10x + 8}_{})$

possible rational zeros:

$\frac{p}{q} \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$

	1	-4	-3	10	8
~~1~~	~~1~~	~~-3~~	~~-6~~	~~4~~	
2	1	-2	-7	-4	0

$f(x) = x^3(x-2)(\underbrace{x^3 - 2x^2 - 7x - 4}_{})$

possible rational zeros:

$\frac{p}{q} \in \{\pm 1, \pm 2, \pm 4\}$

	1	-2	-7	-4
~~2~~	~~1~~	~~0~~		
-1	1	-3	-4	0

$f(x) = x^3(x-2)(x+1)(x^2 - 3x - 4)$

$f(x) = x^3(x-2)(x+1)(x-4)(x+1)$

$f(x) = x^3(x-2)(x+1)^2(x-4)$

x	$-\infty$	-1	0	2	4	∞
$f(x)$	$-\infty$	0 $m=2$	0 $m=3$	0 $m=1$	0 $m=1$	∞

Domain: $x \in \mathbb{R}$

x-in: $x = 0, \quad m = 3$

$x = 2, \quad m = 1$

$x = -1, \quad m = 2$

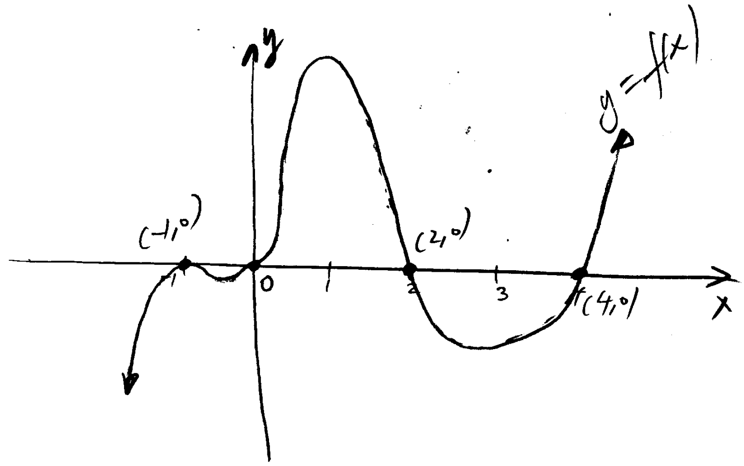
$x = 4, \quad m = 1$

y-in: $(0,0)$

End-behavior is given by x^7

when $x \to \infty, \ y \to \infty$

$x \to -\infty, \ y \to -\infty$



b) $f(x) \geq 0$ iff

$x \in [0, 2] \cup [4, \infty) \cup \{-1\}$

$f(x) < 0$ iff

$x \in (-\infty, -1) \cup (-1, 0) \cup (2, 4)$

(5) (a) $9^{x^2} = 3^{3x+2}$

$(3^2)^{x^2} = 3^{3x+2}$

$3^{2x^2} = 3^{3x+2}$

the exponential function is one-to-one, therefore

$2x^2 = 3x + 2$

$2x^2 - 3x - 2 = 0$

$x = \frac{3 \pm \sqrt{9 + 16}}{4} = \frac{3 \pm 5}{4} \begin{cases} 2 \\ -\frac{1}{2} \end{cases}$

$\boxed{x \in \{2, -\frac{1}{2}\}}$

b) $x^3(4e^{4x}) + 3x^2e^{4x} = 0$

$$\left.\begin{array}{l} x^2e^{4x}(4x+3) = 0 \\ e^{4x} \neq 0, \text{ any } x \end{array}\right\} \Rightarrow$$

$x = 0$ OR $4x+3=0$

$x = -\frac{3}{4}$

$$\boxed{x \in \{0, \tfrac{-3}{4}\}}$$

c) $\ln x = 1 - \ln(x+2)$

Conditions: $\left\{\begin{array}{l} x>0 \\ \text{and} \\ x+2>0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x>0 \\ \text{and} \\ x>-2 \end{array}\right.$

$$\boxed{x>0}$$

$\ln x + \ln(x+2) = 1$

$\ln(x)(x+2) = 1$

$e^1 = x(x+2)$

$x^2 + 2x - e = 0$

$x = \frac{-2 \pm \sqrt{4+4e}}{2}$

$x = \frac{-2 \pm 2\sqrt{e+1}}{2}$

$$\left.\begin{array}{l} x = -1 \pm \sqrt{e+1} \\ \text{but } x>0 \end{array}\right\} \Rightarrow$$

$$\boxed{x = -1 + \sqrt{e+1}}$$

d) $\log x - \log(x+1) = 3\log 4$

Conditions: $\left\{\begin{array}{l} x>0 \\ \text{and} \\ x+1>0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x>0 \\ \text{and} \\ x>-1 \end{array}\right.$

$$\boxed{x>0}$$

$\log \frac{x}{x+1} = \log 4^3$

The logarithmic function is one-to-one $\Rightarrow$

$\frac{x}{x+1} = 64$

$x = 64(x+1)$

$x = 64x + 64$

$-64 = 63x$

$x = -\frac{64}{63}$

but $x>0$, so there are no solutions.

$$\boxed{x \in \emptyset}$$

e) $A = Ba^{ct} + D$ solve for t.

$Ba^{ct} = A - D$

$a^{ct} = \frac{A-D}{B} \quad / \log_a$

$\log_a(a^{ct}) = \log_a\left(\frac{A-D}{B}\right)$

$ct = \log_a\left(\frac{A-D}{B}\right)$

$$\boxed{t = \frac{\log_a\left(\frac{A-D}{B}\right)}{c}}$$

(6) $f(x) = \dfrac{x^2-3x-4}{x^2+x-6}$

$f(x) = \dfrac{(x-4)(x+1)}{(x+3)(x-2)}$

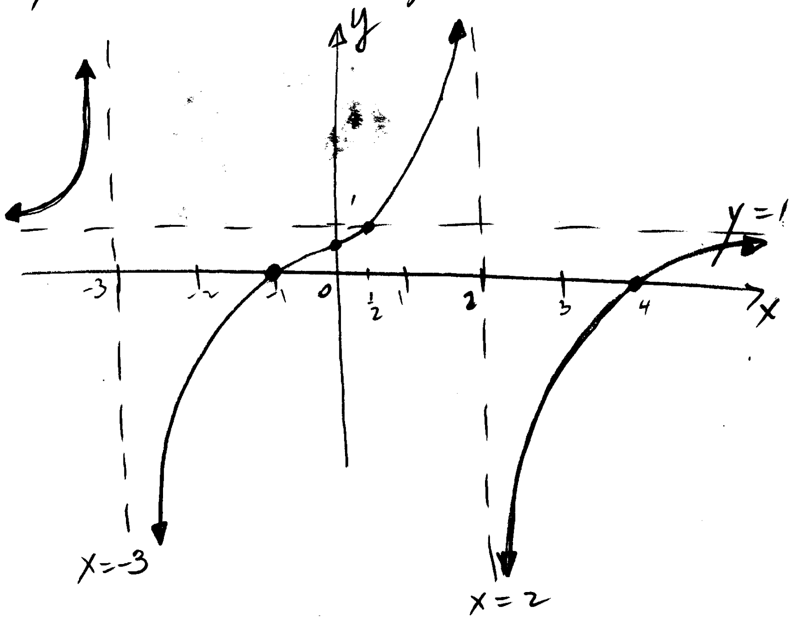
Domain: $x \in \mathbb{R} \setminus \{-3, 2\}$

V. A. $x=-3$, $x=2$

H. A. $y=1$

x-n: $y=0$ iff $(x-4)(x+1)=0$
$x=4$, $x=-1$

y-n: $x=0 \Rightarrow y = \frac{-4}{-6} = \frac{2}{3}$



Intersection of the graph with the line $y=1$ (H.A)

$$\frac{x^2-3x-4}{x^2+x-6} = 1$$

$x^2-3x-4 = x^2+x-6$
$-3x-4 = x-6$
$2 = 4x$, $x = \frac{1}{2}$

So, the graph intersects $y=1$ at $(\frac{1}{2}, 1)$.

Test point: $x=-4$

$f(-4) = \dfrac{(-4-4)(-4+1)}{(-4+3)(-4-2)} = \dfrac{(-)(-)}{(-)(-)}$

$f(-4) > 0$

b) $f(x) \geq 0$ iff
$x \in (-\infty, -3) \cup [-1, 2) \cup [4, \infty)$

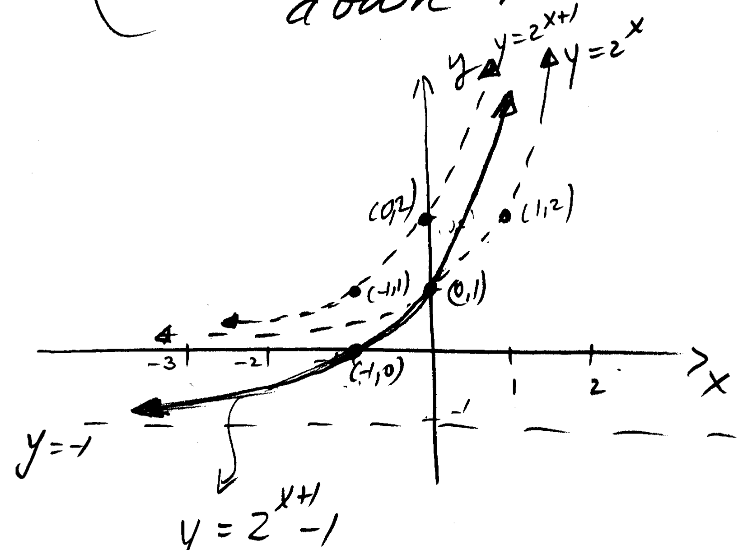
$f(x) < 0$ iff
$x \in (-3, -1) \cup (2, 4)$

(7) $f(x) = 2^{x+1} - 1$

(a) 1st $y = 2^x$

2nd $y = 2^{x+1}$ shift previous graph left 1 unit

3rd $y = 2^{x+1} - 1$, shift previous graph down 1 unit



b) Domain: $x \in \mathbb{R}$

Range: $y \in (-1, \infty)$

H.A. $y = -1$

c) x-∩: from graph $(-1, 0)$

OR

algebraically: let $y = 0$

$2^{x+1} - 1 = 0$

$2^{x+1} = 1$

$x + 1 = 0$

$x = -1$

y-∩: from graph $(0, 1)$

OR

algebraically: at $x = 0$, then $y = 2^{0+1} - 1 = 1$

d) Yes, because f is one-to-one. Any increasing function is one-to-one, and our function is increasing on its entire domain.

1st $y = 2^{x+1} - 1$

and solve for x

$2^{x+1} = y + 1 \quad \Big| \log_2$

$\log_2 2^{x+1} = \log_2 (y+1)$

$x + 1 = \log_2 (y+1)$

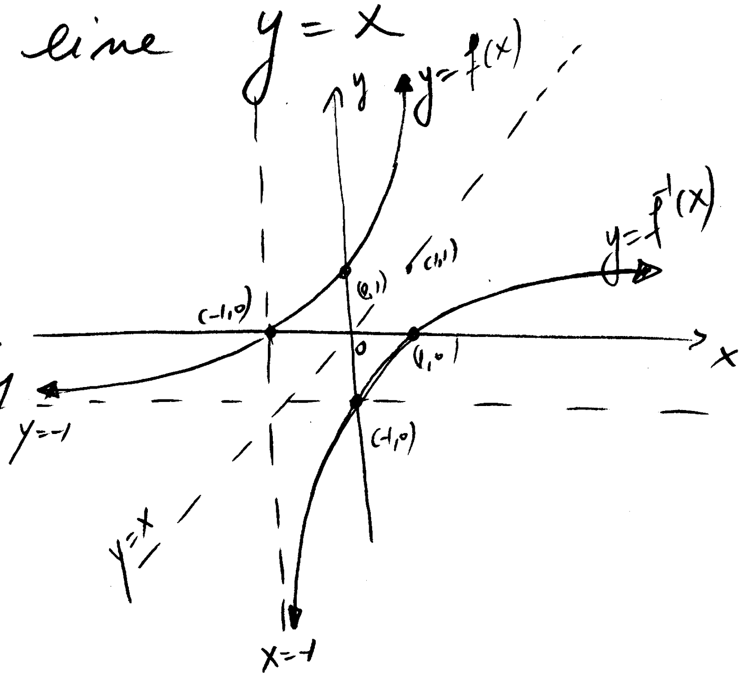
$x = -1 + \log_2 (y+1)$

3rd: $x \leftrightarrow y$

$y = -1 + \log_2 (x+1)$

$$\boxed{f^{-1}(x) = -1 + \log_2 (x+1)}$$

e) The graphs of f and f^{-1} are symmetric about the bisector line $y = x$



x-∩: $(1, 0)$

y-∩: $(-1, 0)$

(8) $h = \dfrac{120}{1+200e^{-0.2t}}$

t = number of years
h = height (in ft)

a) $t = 10$, $h = \dfrac{120}{1+200e^{-0.2(10)}}$

$h = \dfrac{120}{1+200e^{-2}}$ $\boxed{h \approx 4.28 \text{ ft}}$

b) $t = ?$, $h = 50$

$50 = \dfrac{120}{1+200e^{-0.2t}}$

$5\cancel{0}(1+200e^{-0.2t}) = 12\cancel{0}$

$5 + 1000e^{-0.2t} = 12$

$1000e^{-0.2t} = 7$

$e^{-0.2t} = \dfrac{7}{1000}$ | ln

$\ln e^{-0.2t} = \ln \dfrac{7}{1000}$

$-0.2t = \ln \dfrac{7}{1000}$

$t = \dfrac{\ln \frac{7}{1000}}{-0.2}$

$\boxed{t \approx 24.8 \text{ years}}$

(9) Let C = cost (in thousand dollars)
t = year

(b)

t	C
1986	550
1996	1085
2001	2100
2005	2400

(a) C is a function of t because for every input t there is only one output C.

(c) $m_1 = \dfrac{\Delta C}{\Delta t} = \dfrac{1085-550}{1996-1986}$

$m_1 = 53.5$

$m_2 = \dfrac{\Delta C}{\Delta t} = \dfrac{2100-550}{2001-1986}$

$m_2 \approx 103.3$

$m_1 \neq m_2$, so we don't have a linear relationship

(d) Let $(1986, 550)$ and $(2005, 2400)$

$m = \dfrac{\Delta C}{\Delta t} = \dfrac{2400-550}{2005-1986}$

$m = \dfrac{1850}{19}$ $m \approx 97.4$

$y - y_1 = m(x - x_1)$

so, $C - C_1 = m(t - t_1)$

Use $m = 97.4$ and $(1986, 550)$

$C - 550 = 97.4(t - 1986)$

$$\boxed{C = 97.4t - 192{,}886}$$

e) $t = 2009$, then

$C = 97.4(2009) - 192{,}886$

$C = 2790.6$ thousand dollars.

f) C has an inverse because it is a linear equation whose graph is an ascending line, therefore it is one-to-one

$C = 97.4t - 192886$

$97.4t = C + 192886$

$$\boxed{t = \frac{C + 192886}{97.4}}$$

if $f(t) = 97.4t - 192886$ (the cost function)

then $f^{-1}(C) = \dfrac{C + 192886}{97.4}$

The inverse function gives the year when a 30-second TV commercial had a given cost.

$f^{-1}(2200) = \dfrac{2200 + 192886}{97.4}$

$f^{-1}(2200) \approx 2003$

The cost of a TV-commercial was 2200 thousand \$ in 2003

(10) m = # miles driven
C = cost of a taxi ride

a) $C(3.5)$ = cost of riding the taxi for 3.5 miles

if $C(3) = 5.50\$$ and $C(4) = 7\$$,
then $C(3.5) = \frac{5.50 + 7}{2}$
$C(3.5) = 6.25\$$

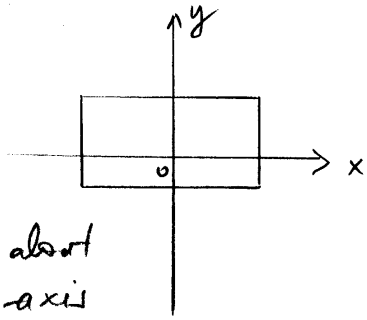
b) $C^{-1}(3.5)$ = the number of miles driven if the cost was 3.5 \$

1st mile costs 2.50 \$
2nd mile costs $4 - 2.50\$ = 1.50\$$
which means
$0.50\$ / 0.33$ mi

So, $3.50\$ = \underbrace{2.50}_{1st\ mi} + 2\underbrace{(0.50)}_{0.33\ mi}$

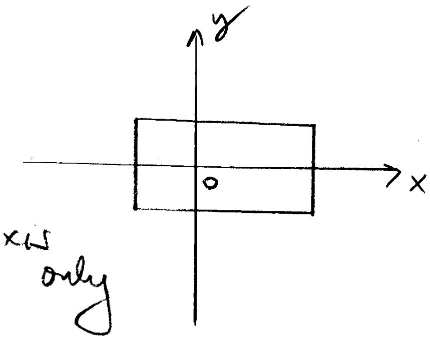
So $C^{-1}(3.5) \approx 1.67$ miles

(11) (a)



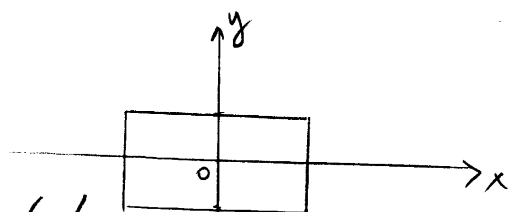
symmetry about the y-axis only

(b)



symmetry about the x-axis only

(c)



symmetry about both axes and the origin

(d)