SOLUTIONS

Math 61 Spring 2007

Name:

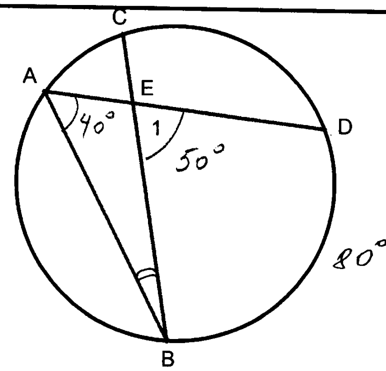
TEST #3 @ 130 points

Write in a neat and organized fashion. Use a straightedge and compass for your drawings.

1. Prove the following theorem (formal proof):
   If two chords in a circle or in congruent circles are congruent, then their arcs are congruent.

2. Prove that the locus of points in a plane equidistant from the sides of an angle is the angle bisector (informal proof).

3. If $m\angle 1 = 50^\circ$ and $m\angle DAB = 40^\circ$, find $m\overset{\frown}{AC}$ and $m\angle ABC$.

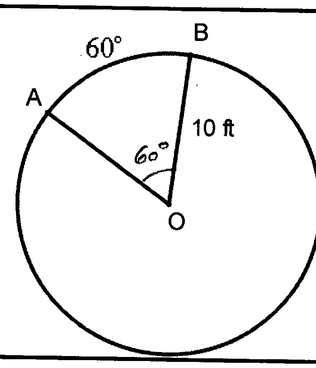


4.
a) Find the circumference of the given circle (exact answer).

b) Find the area of the given circle.

c) Find the length of the arc AB.

d) Find the area of the sector AOB.



5. Sketch a right triangle that has one acute angle θ, and find the other five trigonometric ratios of θ.
   $\sin\theta = \frac{2}{7}$

6. From the top of a 200-ft lighthouse, the angle of depression to a ship in the ocean is 23°. How far is the ship form the base of the lighthouse?

7. Simplify the following expressions:

a) $\frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$

b) $\sin^2\theta\cos\theta + \cos^3\theta$

8. Prove the following trigonometric identities:

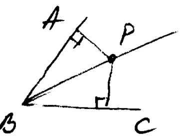
a) $2\tan a \sec a = \frac{1}{1-\sin a} - \frac{1}{1+\sin a}$

b) $(\sin x + \cos x)^2 = 1 + 2\sin x\cos x$

TEST # 3 - SOLUTIONS

(1) Theorem 6.6 section 6.2
- See formal proof in the book.

(2) Proof



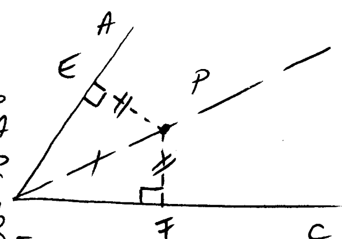
Part I We'll show that if P is a point in plane equidistant from the sides of a given angle, then P is on the bisector of the angle.

Given $\angle ABC$
$d(P, \overrightarrow{BA}) = d(P, \overrightarrow{BC})$

Prove $\overrightarrow{BP}$ = bisector $\angle ABC$

(Condition: $\angle PBA \cong \angle PBC$)

Proof



Let $\overline{PE} \perp \overrightarrow{BA}$, $E \in \overrightarrow{BA}$
$\overline{PF} \perp \overrightarrow{BC}$, $F \in \overrightarrow{BC}$

then $d(P, \overrightarrow{BA}) = PE$
$d(P, \overrightarrow{BC}) = PF$

and $PE = PF$

$\triangle EBP$, $\triangle FBP$ right $\triangle$'s $\begin{cases} \overline{PE} \cong \overline{PF} & \text{(given)} \\ \overline{BP} \cong \overline{BP} & \text{(common side)} \end{cases}$

$\Rightarrow \triangle EBP \cong \triangle FBP$ (HL)

$\Rightarrow \angle PBE \cong \angle PBF$

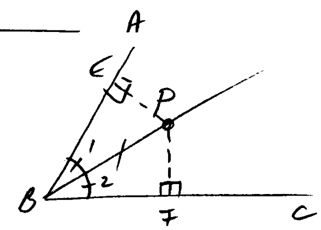
$\Rightarrow \overrightarrow{BP}$ = bisector of $\angle ABC$.

PART II We'll show that if a point P is on the bisector of a given angle, then the point is equidistant from the sides of the angle.

Given $\angle ABC$
$\overrightarrow{BP}$ bisector of $\angle ABC$

Prove $d(P, \overrightarrow{BA}) = d(P, \overrightarrow{BC})$

Proof



$\overrightarrow{BP}$ = bisector $\Rightarrow$
$\angle 1 \cong \angle 2$

Draw $\overline{PE} \perp \overrightarrow{BA}$, $E \in \overrightarrow{BA}$
$\overline{PF} \perp \overrightarrow{BC}$, $F \in \overrightarrow{BC}$

Then $d(P, \overrightarrow{BA}) = PE$
$d(P, \overrightarrow{BC}) = PF$

$\triangle EBP$, $\triangle FBP$ right $\triangle$'s $\begin{cases} \angle 1 \cong \angle 2 & (\overrightarrow{BP} = \text{bisector}) \\ \overline{BP} \cong \overline{BP} & \text{(common side)} \end{cases}$

$\Rightarrow \triangle EBP \cong \triangle FBP$ (HA)

$\Rightarrow \overline{PE} \cong \overline{PF}$

$\Rightarrow d(P, \overrightarrow{BA}) = d(P, \overrightarrow{BC})$

Therefore, a point is equidistant from the sides of a given angle if and only if it is on the bisector of the angle.

(3) $m\angle DAB = 40^\circ$

Also, $m\angle DAB = \frac{1}{2} m\overset{\frown}{BD}$

$40^\circ = \frac{1}{2} m\overset{\frown}{BD} \Rightarrow m\overset{\frown}{BD} = 80^\circ$

$m\angle 1 = 50^\circ$

Also, $m\angle 1 = \frac{1}{2}(m\overset{\frown}{AC} + m\overset{\frown}{BD})$

$50^\circ = \frac{1}{2}(m\overset{\frown}{AC} + 80^\circ)$

$100^\circ = m\overset{\frown}{AC} + 80^\circ$

$20^\circ = m\overset{\frown}{AC}$

$\boxed{m\overset{\frown}{AC} = 20^\circ}$

$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$

$= \frac{1}{2}\, 20^\circ = 10^\circ$

$\boxed{m\angle ABC = 10^\circ}$

(4) Given: $r = 10$ ft

$m\overset{\frown}{AB} = 60^\circ$

a) $C = 2\pi r = 2\pi(10\text{ ft}) = 20\pi$ ft

$\boxed{C = 20\pi \text{ ft}}$

b) $A = \pi r^2 = \pi(10\text{ ft})^2 = 100\pi\ \text{ft}^2$

$\boxed{A = 100\pi\ \text{ft}^2}$

c) $m\angle AOB = m\overset{\frown}{AB} = 60^\circ$

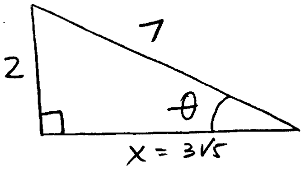
$\dfrac{\ell(\overset{\frown}{AB})}{60^\circ} = \dfrac{2\pi r}{360^\circ} \Rightarrow \ell \overset{\frown}{AB} = \dfrac{2\pi(10\text{ ft})60^\circ}{360^\circ}$

$\boxed{\ell\overset{\frown}{AB} = \frac{10\pi}{3}\text{ ft}}$

d) $\dfrac{A(\text{sector } AOB)}{60^\circ} = \dfrac{\pi r^2}{360^\circ} \Rightarrow A(\text{sector } AOB) = \dfrac{\pi(100\text{ ft}^2)60^\circ}{360^\circ}$

$\boxed{A(\text{sector } AOB) = \frac{50\pi}{3}\ \text{ft}^2}$

(5)



$\sin\theta = \frac{2}{7}$ (given)

$x^2 + 2^2 = 7^2$

$x^2 = 49 - 4 = 45$

$x = \sqrt{45} = 3\sqrt{5}$

$\cos\theta = \dfrac{3\sqrt{5}}{7}$

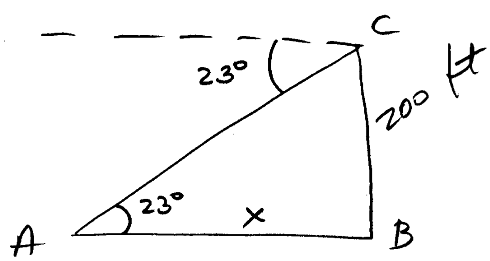
$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{2}{7}}{\frac{3\sqrt{5}}{7}} = \dfrac{2}{3\sqrt{5}} = \dfrac{2\sqrt{5}}{15}$

$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{3\sqrt{5}}{2}$

$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{7}{3\sqrt{5}} = \dfrac{7\sqrt{5}}{15}$

$\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{7}{2}$

(6)



Let x = distance between the ship and the lighthouse

$\triangle ABC$: $\tan 23^\circ = \dfrac{200}{x}$

$x = \dfrac{200}{\tan 23^\circ}$

$x \approx 471$ ft

(7)

(a) $\overset{1+\sin x}{}\quad\overset{\cos x}{}$

$$\frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x} =$$

LCD = $\cos x(1+\sin x)$

$$= \frac{\sin x(1+\sin x) + \cos^2 x}{\cos x(1+\sin x)}$$

$$= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1+\sin x)}$$

$$= \frac{\cancel{\sin x + 1}}{\cos(\cancel{1+\sin x})} = \frac{1}{\cos x}$$

$$= \sec x$$

(b) $\sin^2\theta \cos\theta + \cos^3\theta =$

$$= \cos\theta(\sin^2\theta + \cos^2\theta)$$

$$= \cos\theta$$

(8)

(a) $2\tan a \sec a = \dfrac{1}{1-\sin a} - \dfrac{1}{1+\sin a}$

Proof

$$RHS = \frac{1}{1-\sin a} - \frac{1}{1+\sin a}$$

LCD = $(1-\sin a)(1+\sin a)$

$$= \frac{(1+\sin a)-(1-\sin a)}{(1-\sin a)(1+\sin a)}$$

$$= \frac{\cancel{1}+\sin a-\cancel{1}+\sin a}{1-\sin^2 a} = \frac{2\sin a}{\cos^2 a}$$

$$= \frac{2\sin a}{\cos a \cdot \cos a}$$

$$= 2\tan a \cdot \frac{1}{\cos a}$$

$$= 2\tan a \sec a = LHS$$

Therefore, the given equation is an identity

(b) $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

Proof

$$LHS = (\sin x + \cos x)^2$$

$$= \sin^2 x + 2\sin x\cos x + \cos^2 x$$

$$= \sin^2 x + \cos^2 x + 2\sin x\cos x$$

$$= 1 + 2\sin x \cos x$$

$$= RHS$$

Therefore, the given equation is an identity