Math 150 Spring 07 Name:

# TEST #2 @ 150 points

Solve the problems on separate paper. Clearly label the problems. Show all steps in order to get credit. No proof, no credit given

1. Graph the function $y = \cos x$ . Show the graph over two periods. Answer the following questions:
   a) What is the domain?
   b) What is the range?
   c) What is the period?
   d) What is the amplitude?
   e) What are the x-intercepts?
   f) What is the y-intercept?
   g) Is the function even or odd? How is that shown in the graph?

2. Graph $y = 2\sin\left(x - \frac{\pi}{3}\right)$ over one period. Identify the amplitude , period, and phase shift and label the axes accurately. Explain in words what and how you are graphing.

3. Graph $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1}(x)$ on the same coordinate system, showing the relation between the two graphs (symmetry about the line $y = x$ ). Answer the following questions:
   a) What is the domain and range of $f(x) = \sin x$ ?
   b) What is the domain and range of $f^{-1}(x) = \sin^{-1}(x)$ ?

4. Evaluate the following. Give exact answers whenever possible.
   a) $\sin^{-1}\left(\frac{1}{2}\right)$
   b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
   c) $\tan^{-1}(-1)$
   d) $\cos\left(\sin^{-1}\frac{3}{5}\right)$
   e) $\sin^{-1}\left(\sin\frac{5\pi}{8}\right)$
   f) $\cos^{-1}\left(\cos\frac{2\pi}{7}\right)$
   g) $\tan\left(\tan^{-1}100.23\right)$
   h) $\cos 15^\circ$

5. Prove the following identities:

a) $\dfrac{\sin x+1}{\cos x+\cot x}=\tan x$

b) $\dfrac{\cos t}{1+\sin t}=\dfrac{1-\sin t}{\cos t}$

c) $\tan\left(a+b\right)=\dfrac{\tan a+\tan b}{1-\tan a\tan b}$

6. Solve the following equations.
When appropriate, show EXACT answers.
ONLY when NO exact answer is possible, express solutions rounded to two decimal places.

a) Find ALL solutions: $2\sin x-1=0$

b) Solve on $[0,2p)$: $\sin(2x)=1$

c) Solve on $[0^\circ,360)$: $2\tan q+2=0$

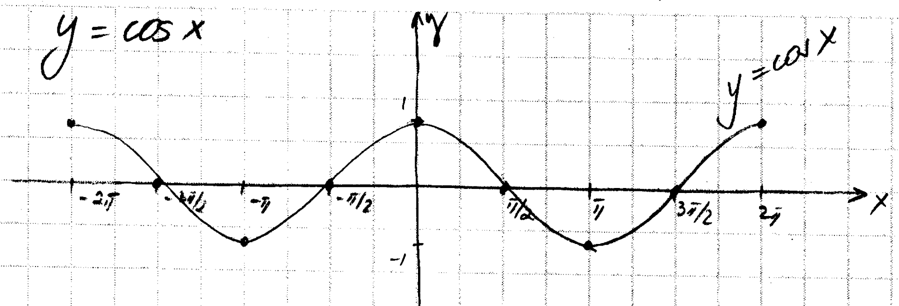
d) Solve on $[0,2p)$: $\cos q=-0.8$

e) Solve on $[0,2p)$: $2\sin^2 x-\sin x-1=0$

f) Find ALL solutions: $\sin 2q-\cos q=0$

g) Solve on $[0,2p)$: $2\sin^2 q-2\cos q-1=0$

① $y = \cos x$




a) Domain: $x \in \mathbb{R}$

b) Range: $y \in [-1, 1]$

c) Period: $T = 2\pi$

d) Amplitude: $A = 1$

e) x-n: $(\pm\frac{\pi}{2}, 0)$, $(\pm\frac{3\pi}{2}, 0)$, $(\pm\frac{5\pi}{2}, 0)$, etc

OR

$((2k+1)\frac{\pi}{2}, 0)$ where $k \in \mathbb{Z}$

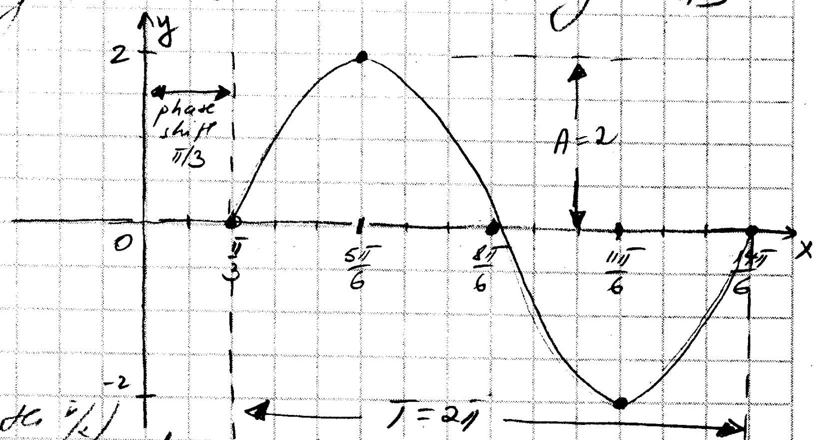
f) y-n: $(0, 1)$

g) $y = \cos x$ is an even function
the graph is symmetric about the y-axis

② $y = 2\sin(x - \frac{\pi}{3})$

amplitude = 2
period $T = 2\pi$
phase shift = $\frac{\pi}{3}$

Take $[\frac{\pi}{3}, \frac{\pi}{3} + 2\pi]$
divide it into 4 equal intervals (each of length $\pi/2$)
then sketch a sine curve of amplitude 2.




$\frac{\pi}{3}$

$\frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$

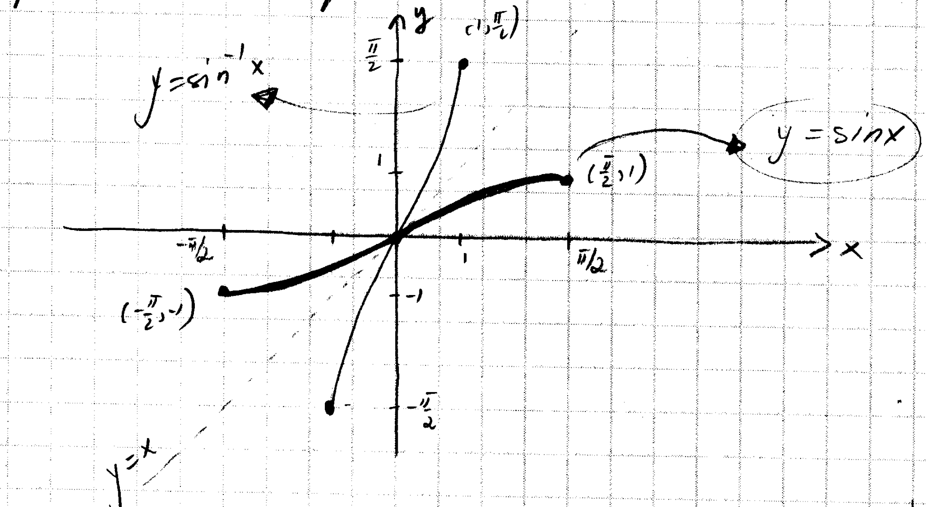
$\frac{5\pi}{6} + \frac{\pi}{2} = \frac{8\pi}{6} = \frac{4\pi}{3}$

$\frac{8\pi}{6} + \frac{\pi}{2} = \frac{11\pi}{6}$

$\frac{11\pi}{6} + \frac{\pi}{2} = \frac{14\pi}{6} = \frac{7\pi}{3}$

(3) $f(x) = \sin x$ $\quad f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$

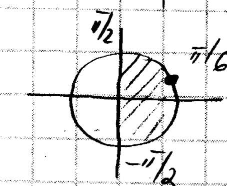
$f^{-1}(x) = \sin^{-1} x$ $\quad f^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$



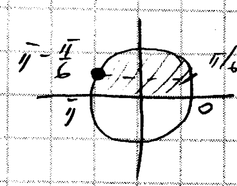

(4) NOTE THAT: $\begin{cases} \sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \cos^{-1} : [-1,1] \to [0, \pi] \\ \tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$

| | $\pi/6$ | $\pi/3$ | $\pi/4$ |
|---|---|---|---|
| sin | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ |
| cos | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ |

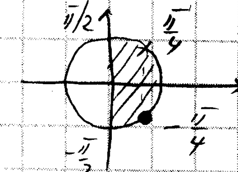
(a) $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$ b/c $\begin{cases} \frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \text{and} \\ \sin \frac{\pi}{6} = \frac{1}{2} \end{cases}$




(b) $\cos^{-1}(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$ b/c $\begin{cases} \frac{5\pi}{6} \in [0, \pi] \\ \text{and} \\ \cos \frac{5\pi}{6} = \frac{-\sqrt{3}}{2} \end{cases}$




(c) $\tan^{-1}(-1) = \frac{-\pi}{4}$ b/c $\begin{cases} -\frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \text{and} \\ \tan(-\frac{\pi}{4}) = -1 \end{cases}$




(d) $\cos(\sin^{-1} \frac{3}{5}) = ?$

let $\sin^{-1} \frac{3}{5} = u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

then $\sin u = \frac{3}{5}$

$\sin^2 u + \cos^2 u = 1$

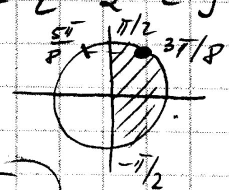
$\left(\frac{3}{5}\right)^2 + \cos^2 u = 1 \Rightarrow \cos^2 u = 1 - \frac{9}{25} = \frac{16}{25}$

$\left.\begin{array}{l} \cos u = \pm \frac{4}{5} \\ \text{but } u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{array}\right\} \Rightarrow \cos u = \frac{4}{5}$

Therefore, $\cos u = \cos\left(\sin^{-1}\frac{3}{5}\right) = \frac{4}{5}$

(e) $\sin^{-1}\left(\sin\frac{5\pi}{8}\right) \neq \frac{5\pi}{8}$ b/c $\frac{5\pi}{8} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\frac{5\pi}{8} = \sin\frac{3\pi}{8}$, $\frac{3\pi}{8} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$




Therefore,

$\sin^{-1}\left(\sin\frac{5\pi}{8}\right) = \sin^{-1}\left(\sin\frac{3\pi}{8}\right) = \frac{3\pi}{8}$

(f) $\cos^{-1}\left(\cos\frac{2\pi}{7}\right) = \frac{2\pi}{7}$ b/c $\frac{2\pi}{7} \in [0, \pi]$

(g) $\tan(\tan^{-1} 100.23) = 100.23$ b/c $100.23 \in \mathbb{R}$

(h) $\cos 15° = ?$

Method I using $\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\cos 15° = \cos(45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

Method II using $\cos 2a = 2\cos^2 a - 1$

$\cos^2 a = \frac{1+\cos 2a}{2} \Rightarrow \cos a = \pm\sqrt{\frac{1+\cos 2a}{2}}$

$\cos 15° = \sqrt{\frac{1+\cos 2(15°)}{2}}$ $15° \in I$, therefore $\cos 15° > 0$

$\cos 15° = \sqrt{\frac{1+\cos 30°}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$

Note that $\frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{2+\sqrt{3}}}{2} \iff (\sqrt{6}+\sqrt{2})^2 = 4(2+\sqrt{3})$

(method I) (method II) $\iff 6+2+2\sqrt{12} = 8+4\sqrt{3}$

$\iff 8+4\sqrt{3} = 8+4\sqrt{3}$

(5) (a) $\dfrac{\sin x+1}{\cos x+\cot x} = \tan x$ //

Proof

$$LHS = \frac{\sin x+1}{\cos x+\cot x}$$

$$= \frac{\sin x+1}{\cos x+\frac{\cos x}{\sin x}}$$

$$= \frac{\sin x(\sin x+1)}{\cos x(\sin x+1)}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x = RHS$$

(b) $\dfrac{\cos t}{1+\sin t} = \dfrac{1-\sin t}{\cos t}$ //

Proof

$$\frac{\cos t}{1+\sin t} = \frac{1-\sin t}{\cos t}$$

$$\Longleftrightarrow$$

$$\cos^2 t = (1-\sin t)(1+\sin t)$$

$$\Longleftrightarrow$$

$$\cos^2 t = 1-\sin^2 t$$

$$\Longleftrightarrow$$

$$\cos^2 t = \cos^2 t \quad \text{true}$$

Therefore, $\dfrac{\cos t}{1+\sin t} = \dfrac{1-\sin t}{\cos t}$

(c) $\tan(a+b) = \dfrac{\tan a+\tan b}{1-\tan a\tan b}$ //

Proof

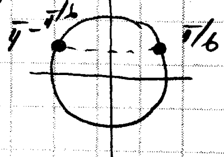
$$LHS = \tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)}$$

$$= \frac{\sin a\cos b+\sin b\cos a}{\cos a\cos b-\sin a\sin b}$$

$$= \frac{\frac{\sin a\cos b}{\cos a\cos b}+\frac{\sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b}{\cos a\cos b}-\frac{\sin a\sin b}{\cos a\cos b}}$$

$$= \frac{\tan a+\tan b}{1-\tan a\tan b} = RHS$$

(6) (a) $2\sin x - 1 = 0$



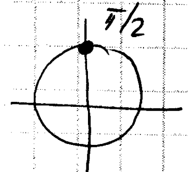

| | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{4}$ |
|---|---|---|---|
| sin | $1/2$ | $\sqrt{3}/2$ | $\sqrt{2}/2$ |
| cos | $\sqrt{3}/2$ | $1/2$ | $\sqrt{2}/2$ |
| tan | $\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ | $1$ |

$\sin x = \frac{1}{2}$

Note: $T = 2\pi$

$$\begin{cases} x = \frac{\pi}{6} + 2k\pi \\ \text{OR} \\ x = \frac{5\pi}{6} + 2k\pi \end{cases}, k \in \mathbb{Z}$$

(b) $\sin 2x = 1$




Note: $T = 2\pi$

$2x = \frac{\pi}{2} + 2k\pi$

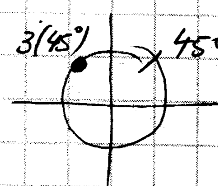
$x = \frac{\pi}{4} + k\pi, \; k \in \mathbb{Z}$

If $k = 0 \Rightarrow x = \frac{\pi}{4}$

$k = 1 \Rightarrow x = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

Solution set $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$

(c) $2\tan\theta + 2 = 0$




$\tan\theta = -1$

Note: $T = 180°$

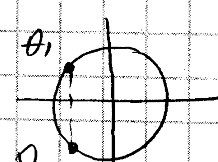
$\theta = 135° + 180°k, \; k \in \mathbb{Z}$

If $k = 0 \Rightarrow \theta = 135°$

If $k = 1 \Rightarrow \theta = 135° + 180°$
$= 315°$

Solution set $\{135°, 315°\}$

(d) $\cos\theta = -0.8$




$\theta_1 = \cos^{-1}(-0.8) \in [0, \pi]$

$\theta_1 \approx 2.50$.

Then, $\theta_2 = 2\pi - \theta_1$
$= 2\pi - \cos^{-1}(-0.8)$

$\theta_2 \approx 3.79$

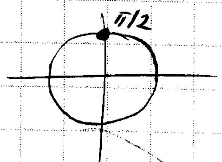
Solution set
$\{2.50, 3.79\}$

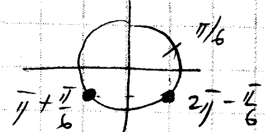
(e) $2\sin^2 x - \sin x - 1 = 0$

$\sin x = \dfrac{1 \pm \sqrt{1+8}}{4}$

$= \dfrac{1 \pm 3}{4}$

$\sin x = 1$ OR $\sin x = \dfrac{-1}{2}$




$x = \dfrac{\pi}{2}$

$x = \dfrac{7\pi}{6}$ OR $x = \dfrac{11\pi}{6}$

Solution set:

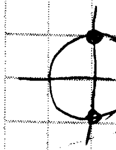
$\left\{ \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6} \right\}$

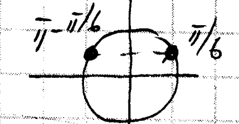
(f) $\sin 2\theta - \cos\theta = 0$

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ OR $\sin\theta = \dfrac{1}{2}$






$\theta = \dfrac{\pi}{2} + 2k\pi$ OR $\theta = \dfrac{3\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

$\theta = \dfrac{\pi}{6} + 2k\pi$ OR $\theta = \dfrac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$

(g) $2\sin^2\theta - 2\cos\theta - 1 = 0$

$2(1 - \cos^2\theta) - 2\cos\theta - 1 = 0$

$2 - 2\cos^2\theta - 2\cos\theta - 1 = 0$

$-2\cos^2\theta - 2\cos\theta + 1 = 0$

$2\cos^2\theta + 2\cos\theta - 1 = 0$

$\cos\theta = \dfrac{-2 \pm \sqrt{4+8}}{4}$

$= \dfrac{-2 \pm 2\sqrt{3}}{4}$

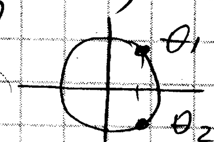
$\cos\theta = \dfrac{-1 \pm \sqrt{3}}{2}$

$\cos\theta = \dfrac{-1+\sqrt{3}}{2}$ OR $\cos\theta = \dfrac{-1-\sqrt{3}}{2} < -1$

$\cos\theta \approx 0.366$ — not possible, No solution

$\theta_1 = \cos^{-1}(0.366) \in [0, \pi]$

$\theta_1 \approx 1.19$




then

$\theta_2 = 2\pi - \theta_1$

$= 2\pi - 1.19$

$\theta_2 \approx 5.09$

Solution set $\{1.19, 5.09\}$