① Find the exact value of each expression (if it is defined).

(a) $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$ b/c $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\pi/3 \in [-\pi/2, \pi/2]$

(b) $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{-\pi}{4}$ b/c $\sin\left(\frac{-\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

$-\frac{\pi}{4} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

(c) $\tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ b/c $\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$\frac{\pi}{6} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(d) $\tan^{-1} 0 = 0$ b/c $\tan 0 = 0$

$0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(e) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ b/c $\cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2}$

$\frac{5\pi}{6} \in [0, \pi]$

(f) $\sin(\sin^{-1}\frac{1}{3}) = \frac{1}{3}$ b/c $\frac{1}{3} \in [-1, 1]$

(g) $\cos^{-1}(\cos\frac{\pi}{3}) = \frac{\pi}{3}$ b/c $\frac{\pi}{3} \in [0, \pi]$

(h) $\sin^{-1}(\sin\frac{5\pi}{6}) \neq \frac{5\pi}{6}$ b/c $\frac{5\pi}{6} \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin\frac{5\pi}{6} = \sin\frac{\pi}{6}$

$\sin^{-1}(\sin\frac{5\pi}{6}) = \sin^{-1}(\sin\frac{\pi}{6}) = \frac{\pi}{6}$

$\pi/6 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

(i) $\cos(\sin^{-1}\frac{\sqrt{3}}{2}) = ?$

let $\sin^{-1}\frac{\sqrt{3}}{2} = u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

then $\sin u = \frac{\sqrt{3}}{2}$

$\cos^2 u = 1 - \sin^2 u$

$= 1 - \frac{3}{4} = \frac{1}{4}$

$\left.\begin{array}{l}\cos u = \pm\frac{1}{2} \\ \text{but } u \in [-\frac{\pi}{2}, \frac{\pi}{2}]\end{array}\right\} \Rightarrow \cos u = \frac{1}{2}$

Therefore, $\cos(\sin^{-1}\frac{\sqrt{3}}{2}) = \frac{1}{2}$

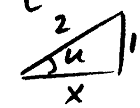
(j) $\tan(\sin^{-1}\frac{1}{2}) = \tan\left(\frac{\pi}{6}\right)$

$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

OR

let $\sin^{-1}\frac{1}{2} = u \in [-\pi/2, \pi/2]$

then $\sin u = \frac{1}{2}$



then $x^2 = 4 - 1 = 3$

$x = \pm\sqrt{3}$

$\cos u = \frac{\sqrt{3}}{2}$ ($\cos u > 0$ b/c $u \in [-\pi/2, \pi/2]$)

$\tan u = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

So $\tan(\sin^{-1}\frac{1}{2}) = \frac{\sqrt{3}}{3}$

(k) $\tan^{-1}(\tan\frac{2\pi}{3}) \neq \frac{2\pi}{3}$ b/c

$\frac{2\pi}{3} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$

$\tan\frac{2\pi}{3} = \tan\frac{-\pi}{3}$

$\tan^{-1}(\tan\frac{2\pi}{3}) = \tan^{-1}(\tan\frac{-\pi}{3})$

$= \frac{-\pi}{3} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

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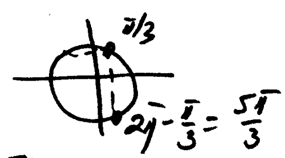
② Solve the equations in $\mathbb{R}$

(a) $2\cos x - 1 = 0$

$2\cos x = 1$

$\cos x = \frac{1}{2}$



$$\begin{cases} x = \frac{\pi}{3} + 2k\pi, \ k \in \mathbb{Z} \\ \text{OR} \\ x = \frac{5\pi}{3} + 2k\pi, \ k \in \mathbb{Z} \end{cases}$$

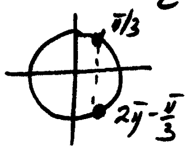
(b) $4\cos^2 x - 1 = 0$ in $\mathbb{R}$

$4\cos^2 x = 1$

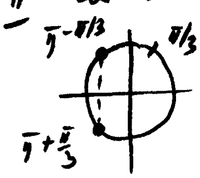
$\cos^2 x = \frac{1}{4}$

$\cos x = \pm \frac{1}{2}$

I $\cos x = \frac{1}{2}$ $\Rightarrow \begin{cases} x = \frac{\pi}{3} + 2k\pi \\ \text{OR} \\ x = \frac{5\pi}{3} + 2k\pi \end{cases}$ $k \in \mathbb{Z}$



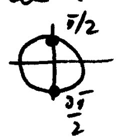
II $\cos x = \frac{-1}{2}$ $\Rightarrow \begin{cases} x = \frac{2\pi}{3} + 2k\pi \\ \text{OR} \\ x = \frac{4\pi}{3} + 2k\pi \end{cases}$ $k \in \mathbb{Z}$

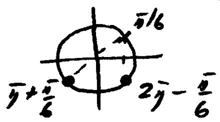


(c) $\cos x(2\sin x + 1) = 0$ in $[0, 2\pi)$

I $\cos x = 0$ OR II $2\sin x = -1$

$\sin x = \frac{-1}{2}$





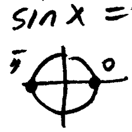
$\begin{cases} x = \frac{\pi}{2} \text{ OR} \\ x = \frac{3\pi}{2} \end{cases}$ $\begin{cases} x = \frac{7\pi}{6} \text{ OR} \\ x = \frac{11\pi}{6} \end{cases}$

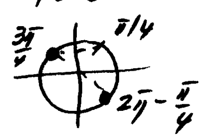
$x \in \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\}$

(d) $\tan x \sin x + \sin x = 0$ in $[0, 2\pi)$

$\sin x(\tan x + 1) = 0$

I $\sin x = 0$ OR II $\tan x = -1$





$\begin{cases} x = 0 \text{ OR} \\ x = \pi \end{cases}$ $\begin{cases} x = \frac{3\pi}{4} \text{ OR} \\ x = \frac{7\pi}{4} \end{cases}$

$x \in \{0, \pi, \frac{3\pi}{4}, \frac{7\pi}{4}\}$

e) $\sin^2 x = 4 - 2\cos^2 x$ in $\mathbb{R}$

$(1 - \cos^2 x) = 4 - 2\cos^2 x$

$-\cos^2 x + 2\cos^2 x = 4 - 1$

$\cos^2 x = 3$

I $\cos x = \sqrt{3} > 1$ OR II $\cos x = -\sqrt{3} < -1$

$x \in \emptyset$ $x \in \emptyset$

Therefore, $x \in \emptyset$
(no solutions)

f) $\sqrt{3}\sin 2x = \cos 2x$ in $[0, 2\pi)$

Note that $\cos 2x \neq 0$

(if $\cos 2x = 0$, then $\sqrt{3}\sin 2X = 0$
$\sin 2X = 0$
but sin a and cos a can't be zero in the same time)

Therefore, because $\cos 2x \neq 0$ we can divide both sides of the equation by $\cos 2x \neq 0$

$\sqrt{3}\tan 2x = 1$

$\tan 2x = \frac{1}{\sqrt{3}}$

$2x = \frac{\pi}{6} + k\pi$ ($T = \pi$ for the tangent function)

$x = \frac{\pi}{12} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$

if $k = 0$, $x = \frac{\pi}{12}$

$k = 1$, $x = \frac{\pi}{12} + \frac{\pi}{2} = \frac{7\pi}{12}$

$k = 2$, $x = \frac{\pi}{12} + \pi = \frac{13\pi}{12}$

$k = 3$, $x = \frac{\pi}{12} + \frac{3\pi}{2} = \frac{19\pi}{12}$

$k = 4$, $x = \frac{\pi}{12} + 2\pi > 2\pi$

$x \in \{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}\}$

(g) $\cos^2 2x - \sin^2 2x = 0$

Method I

$$\cos^2 2x = \sin^2 2x$$

Note that $\cos^2 2x \neq 0$

(if $\cos 2x = 0$, then $\sin 2x = 0$ which is impossible; sin and cos cannot be zero in the same time)

So, because $\cos 2x \neq 0$, we can divide both sides of the equation by $\cos^2 2x \neq 0$

$$\frac{\cos^2 2x}{\cos^2 2x} = \frac{\sin^2 2x}{\cos^2 2x}$$

$$\tan^2 2x = 1$$

I $\tan 2x = 1$ OR II $\tan 2x = -1$

$2x = \frac{\pi}{4} + k\pi$ | $2x = \frac{3\pi}{4} + k\pi$

($T = \pi$ for the tangent function)

$x = \frac{\pi}{8} + \frac{k\pi}{2}$ (*) $x = \frac{3\pi}{8} + \frac{k\pi}{2}$

$k=0,\ x = \frac{\pi}{8}$	$k=0,\ x = \frac{3\pi}{8}$
$k=1,\ x = \frac{\pi}{8} + \frac{\pi}{2} = \frac{5\pi}{8}$	$k=1,\ x = \frac{3\pi}{8} + \frac{\pi}{2} = \frac{7\pi}{8}$
$k=2,\ x = \frac{\pi}{8} + \pi = \frac{9\pi}{8}$	$k=2,\ x = \frac{3\pi}{8} + \pi = \frac{11\pi}{8}$
$k=3,\ x = \frac{\pi}{8} + \frac{3\pi}{2} = \frac{13\pi}{8}$	$k=3,\ x = \frac{3\pi}{8} + \frac{3\pi}{2} = \frac{15\pi}{8}$
$k=4,\ x = \frac{\pi}{8} + 2\pi > 2\pi$	$k=4,\ x = \frac{3\pi}{8} + 2\pi > 2\pi$

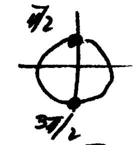
So, $x \in \left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$

Method II

$$\cos^2 2x - \sin^2 2x = 0$$

We know that $\cos 2a = \cos^2 a - \sin^2 a$

Therefore, $\cos^2 2x - \sin^2 2x = \cos 2(2x)$
$= \cos 4x$



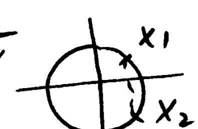
$$\cos 4x = 0$$

I $4x = \frac{\pi}{2} + 2k\pi$ OR II $4x = \frac{3\pi}{2} + 2k\pi$

$x = \frac{\pi}{8} + \frac{k\pi}{2}$ OR $x = \frac{3\pi}{8} + \frac{k\pi}{2}$

Same as before (*)

(h) $\cos x = 0.4$ in $\mathbb{R}$



$x_1 = \cos^{-1} 0.4 + 2k\pi$

$x_1 \approx 1.16 + 2k\pi$

Then, $x_2 = 2\pi - x_1$

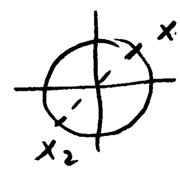
$x_2 = 2\pi - \cos^{-1} 0.4 + 2k\pi$

$x_2 \approx 5.12 + 2k\pi$

Therefore, $\begin{cases} x_1 = \cos^{-1} 0.4 + 2k\pi \\ \text{OR} \\ x_2 = 2\pi - \cos^{-1} 0.4 + 2k\pi \\ k \in \mathbb{Z} \end{cases}$

(i) $2\tan x = 13$ in $\mathbb{R}$

$\tan x = \frac{13}{2}$



$x = \tan^{-1} \frac{13}{2} + k\pi$

$x \approx 1.42 + k\pi$

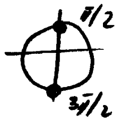
(j) $2\sin(2x) - \cos x = 0$ in $[0, 2\pi)$

$2(2\sin x \cos x) - \cos x = 0$

$4\sin x \cos x - \cos x = 0$

$\cos x\,(4\sin x - 1) = 0$

I $\cos x = 0$ OR II $4\sin x - 1 = 0$

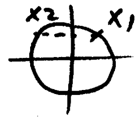


$4\sin x = 1$

$\sin x = \frac{1}{4}$

$\begin{cases} x = \frac{\pi}{2} \text{ OR} \\ x = \frac{3\pi}{2} \end{cases}$

$x_1 = \sin^{-1}\frac{1}{4}$



$\begin{cases} x_1 \approx 0.25 \\ \text{OR } x_2 = \pi - x_1 \\ x_2 = \pi - \sin^{-1}\frac{1}{4} \end{cases}$

$x_2 \approx 2.89$

$x \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \sin^{-1}\frac{1}{4}, \pi - \sin^{-1}\frac{1}{4}\right\}$

(k) $\cos x \cos 3x - \sin x \sin 3x = 0$ in $[0, 2\pi)$

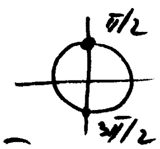
We know that

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

Therefore,

$\cos x \cos 3x - \sin x \sin 3x = \cos(x+3x)$
$= \cos 4x$

So, $\cos 4x = 0$



I $4x = \frac{\pi}{2} + 2k\pi$ OR II $4x = \frac{3\pi}{2} + 2k\pi$

$x = \frac{\pi}{8} + \frac{k\pi}{2}$

$k=0,\ x = \frac{\pi}{8}$

$k=1,\ x = \frac{5\pi}{8}$

$k=2,\ x = \frac{9\pi}{8}$

$k=3,\ x = 13\pi/8$

$x = \frac{3\pi}{8} + \frac{k\pi}{2}$

$k=0,\ x = 3\pi/8$

$k=1,\ x = 7\pi/8$

$k=2,\ x = \frac{11\pi}{8}$

$k=3,\ x = \frac{15\pi}{8}$

Therefore,

$x \in \left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$

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(3) (a) $\cos 2x = 2\cos x$ is a false statement

$\cos 2x = \cos^2 x - \sin^2 x$

For example, if $x = 0$, $\cos 2x = 1$

$2\cos x = 2$

and $1 \neq 2$

(b) $\sin a + \sin b = \sin(a+b)$ is a false statement

$\sin(a+b) = \sin a \cos b + \sin b \cos a$

For example, if $a = \frac{\pi}{2} = b$,

then $\sin a + \sin b = 1 + 1 = 2$

$\sin(a+b) = \sin(\pi) = 0$

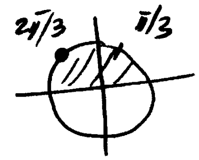
and $2 \neq 0$

(c) $\sin^{-1}\frac{2}{3} \approx 0.72973$ true

$\sin^{-1}: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\frac{2}{3} \in [-1,1]$, $0.72973 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

(d) $\cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$



true

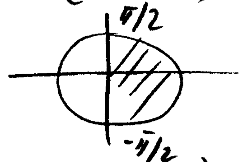
$\cos^{-1}: [-1,1] \to [0, \pi]$

$-\frac{1}{2} \in [-1,1]$, $\frac{2\pi}{3} \in [0, \pi]$

and $\cos\frac{2\pi}{3} = \frac{-1}{2}$

(e) $\tan^{-1}(-1) = \frac{3\pi}{4}$ false

$\tan^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



and $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan^{-1}(-1) = -\frac{\pi}{4}$

(f) $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$ OR

$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{2\pi}{3}$

false b/c

$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$ only!

$\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(g) $\cos^{-1}(-2)$ is not defined

true

$\cos^{-1} : [-1,1] \to [0,\pi]$

and $-2 \notin [-1,1]$