Circles
Sections 6.1 & 6.2

The many practical uses of the circle range from the wheel to the near-circular orbits of some communication satellites. The mechanical uses of the circle have been known for thousands of years, and the ancient Greeks contributed significantly to our understanding of the circle's mathematical properties. The full moon, ripples in a pond when a stone is dropped in, and the shape of some bird's nests show some of the circles that appear in nature.

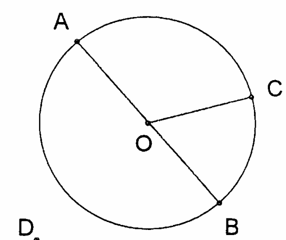
Our study of circles begins with some definitions, an explanation of the standard symbols used, and certain figures related to circles.

Definition A **circle** is the set of all points in a plane that are at a given distance from a given point in the plane.

The given distance = radius $OA = OC = OB = r$

The given point = center O

Notation: $\odot O$ - the circle with center O



Note: A circle divides the plane into three distinct subsets:

- the interior $O \in$ int $\odot O$
- the circle itself $A, B, C \in \odot O$
- the exterior $D \in$ ext $\odot O$

Note: The radius of a circle is defined above as a number. It is standard practice, however, for "radius" to also mean a line segment, as in the following definition. You can usually determine which meaning of the word " radius" is intended by the context in which it is used.

Definition A **radius** of a circle is a segment that joins the center of the circle to a point on the circle. $\overline{OA}$

Definition A **diameter** of a circle is a segment whose endpoints are points of the circle and it contains the center of the circle. $\overline{AB}$

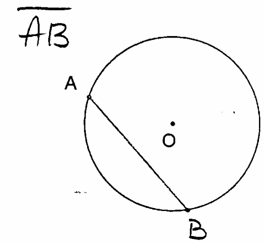
Theorem In any given circle all radii are congruent and all diameters are congruent.
(radii $\odot \cong$ and diams $\odot \cong$)

Definition Two or more **circles are congruent** if they have congruent radii
($\odot s \cong$ iff radii $\cong$).

Definition Two or more coplanar **circles are concentric** if they have the same center

Question: How many circles can share the same center? infinitely many

Definition A line segment is a **chord** of a circle if its endpoints are points of the circle. $\overline{AB}$



Questions: 1) Is a diameter a chord? yes

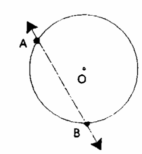
2) Is a radius of chord? No

3) What is the longest possible chord? the diameter

4) How is the length of a chord related to its distance from the center? the closer to the center, the longer the chord

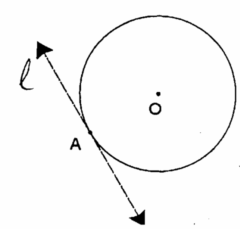
Definition A line (or segment or ray) is a **secant** if it intersects a circle at exactly twp points.

$\overleftrightarrow{AB}$ - secant $\overleftrightarrow{AB} \cap \odot O = \{A, B\}$

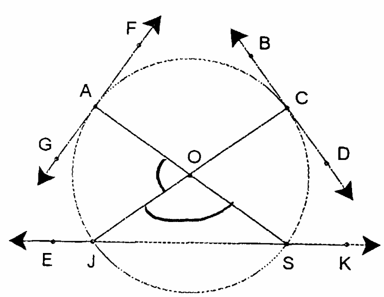


Definition A line is a **tangent** to a circle if it intersects the circle at exactly one point.

l - tangent $l \cap \odot O = \{A\}$



Problem #1 In the given figure, name:



a) fours radii $\overline{OA}, \overline{OB}, \overline{OS}, \overline{OJ}$

b) two diameters $\overline{AS}, \overline{CJ}$

c) three chords $\overline{JS}, \overline{JC}, \overline{AS}$

d) two tangents $\overleftrightarrow{GF}, \overleftrightarrow{BD}$

e) one secant $\overleftrightarrow{EK}$

Several types of angles associated with circles are seen in the above figure. The next definition describes the most fundamental of these angles.

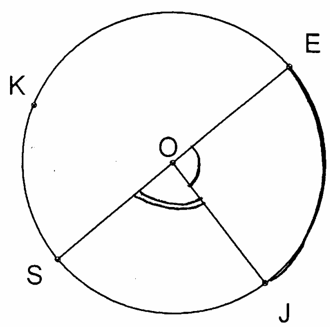
Definition An angle is a **central angle** of a circle if its vertex is the center of the circle

A central angle may be - acute $\angle AOJ$

- right

- obtuse (measure less than 180°) $\angle JOS$

These angles "cut off" portions of the circle called arcs.



Definition A **minor arc** is the set of points of a circle that are on a central angle or in its interior.

Example: $\overset{\frown}{EJ}$ (corresponding to $\angle EOJ$)

Definition The **intercepted arc of an angle** is the minor arc associated with the central angle.

Example: What is the intercepted arc of $\angle SOJ$? $\overset{\frown}{SJ}$

Definition A **major arc** is the set of points of a circle that are on a central angle or in its exterior.

Example: $\overset{\frown}{EKJ}$ (corresponding to $\angle EOJ$)

Definition A **semicircle** is the set of points of a circle that are on, or are on one side of, a line containing a diameter

Example: $\overset{\frown}{SKE}$, $\overset{\frown}{SJE}$

Definition The degree measure of

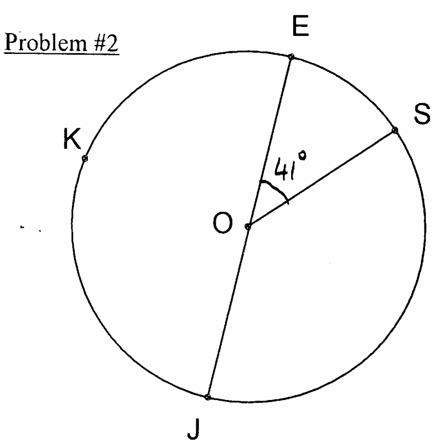
a) a minor arc is the measure of its central angle (also known as **The Central Angle Postulate**),

b) a semicircle is 180°,

c) a circle is 360°,

d) a major arc is 360° minus the measure of its associated minor arc.

Problem #2



Given: $\odot O$

$m\angle EOS = 41^\circ$

Find:

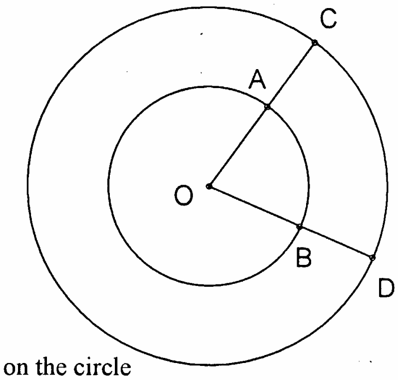
$m\overset{\frown}{ES}$ — $m\overset{\frown}{ES} = m\angle EOS = 41^\circ$

$m\overset{\frown}{ESJ}$ — $m\overset{\frown}{ESJ} = 180^\circ$

$m\angle SOJ$ — $m\angle SOJ = 180^\circ - 41^\circ = 139^\circ$

$m\overset{\frown}{SJ}$ — $m\overset{\frown}{SJ} = m\angle SOJ = 139^\circ$

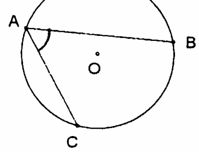
$m\overset{\frown}{EKS}$ — $m\overset{\frown}{EKS} = 360^\circ - 41^\circ = 319^\circ$



Note: The degree measure of an arc is not a measure of the arc's length.

For the concentric circles in the figure, $m\overset{\frown}{AB} = m\overset{\frown}{CD}$ because the arcs have the same central angle, but certainly $\overset{\frown}{AB}$ is not as long as $\overset{\frown}{CD}$.

Definition An angle is an **inscribed angle** of a circle if its vertex is a point on the circle and its sides are chords of the circle.



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Central Angles, Arcs, and Chords

There are some important properties about central angles, arcs, and chords that are associated with a given circle or with two circles that are the same size. But what is meant by "the same size"? congruent circles

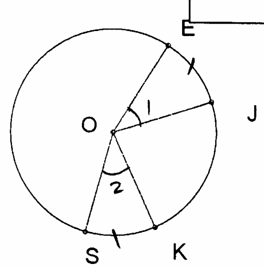
Definition Two **arcs** of a circle or of congruent circles are congruent iff their degree measures are equal.

Note: Since congruent arcs are defined in terms of numbers (degree measures), the addition, subtraction, multiplication, and division properties of congruence may be easily extended to include congruence between arcs.

Theorems relating central angles, arcs, and chords in the same or congruent circles

Theorem 1 (6.1 – T 6.1.3)

If two minor arcs of a circle or of congruent circles are congruent, then their central angles are congruent (if $\overset{\frown}{}s \cong$, central $\angle s \cong$).



Given ⊙O
$\overset{\frown}{EJ} \cong \overset{\frown}{SK}$

Prove $\angle 1 \cong \angle 2$

Proof

Statements	Reasons
1. ⊙O, $\overset{\frown}{EJ} \cong \overset{\frown}{SK}$	1. given
2. $m\overset{\frown}{EJ} = m\overset{\frown}{SK}$	2. definition $\cong$ ⌒'s
3. $m\angle 1 = m\overset{\frown}{EJ}$	3. Central ∡ Postulate
4. $m\overset{\frown}{SK} = m\angle 2$	4. Central ∡ Postulate
5 (2,3,4) $m\angle 1 = m\angle 2$	5. Transitivity
6. $\angle 1 \cong \angle 2$	6. definition $\cong$ <s

Converse 1 (6.1 – T 6.1.4)

(Converse of Theorem 1)

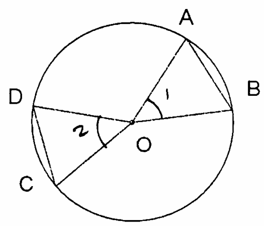
if two central angles in a circle or in congruent circles are congruent, then their arcs are congruent.
(if central $\angle s \cong$, $\overset{\frown}{}s \cong$)

Theorem 2 If two central angles in a circle or in congruent circles are congruent, then their chords are congruent (if central $\angle s \cong$, chords $\cong$).

Write a formal proof.

Given $\odot O$
$\angle 1 \cong \angle 2$

Prove $\overline{AB} \cong \overline{CD}$



Proof

Statements	Reasons
1. $\odot O$, $\angle 1 \cong \angle 2$	1. given
2. $\triangle AOB$, $\triangle COD$: $\overline{AO} \cong \overline{DO}$, $\angle 1 \cong \angle 2$, $\overline{BO} \cong \overline{CO}$	2. radii $\odot$, given, radii $\odot$
3. $\triangle AOB \cong \triangle COD$	3. SAS
4. $\overline{AB} \cong \overline{CD}$	4. CPCTC

Converse 2 (Converse of Theorem 2)

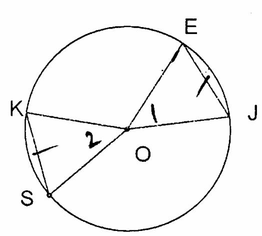
If two chords in a circle or congruent circles are congruent then their central angles are congruent
(if chords $\cong$, central $\angle s \cong$).

Theorem 3 (6.1 – T 6.1.5) If two chords in a circle or in congruent circles are congruent, then their arcs are congruent (if chords $\cong$, $\overset{\frown}{s} \cong$).

Write a formal proof.

Given $\odot O$
$\overline{KS} \cong \overline{EJ}$

Prove $\overset{\frown}{KS} \cong \overset{\frown}{EJ}$



Proof

Statements	Reasons
1. $\odot O$, $\overline{KS} \cong \overline{EJ}$	1. given
2. Draw radii $\overline{OE}$, $\overline{OJ}$, $\overline{OK}$, $\overline{OS}$	2. two points determine a line
3. $\triangle EOJ$, $\triangle KOS$: $\overline{EO} \cong \overline{KO}$, $\overline{JO} \cong \overline{SO}$, $\overline{EJ} \cong \overline{KS}$	3. radii $\odot$, radii $\odot$, given
4. $\triangle EOJ \cong \triangle KOS$	4. SSS
5. $\angle 1 \cong \angle 2$	5. CPCTC
6. $\overset{\frown}{EJ} \cong \overset{\frown}{KS}$	6. Central $\angle$ Postulate

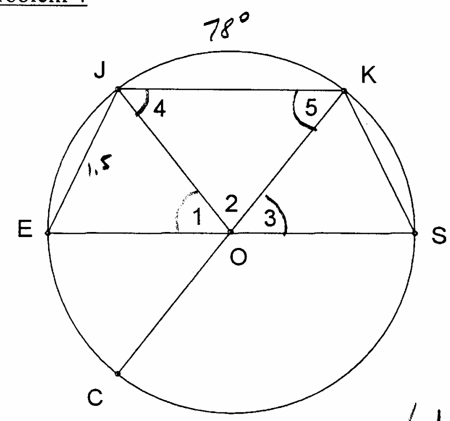
Converse 3 (Converse of Theorem 3)
(6.1 – T. 6.1.6)

If two arcs in a circle or congruent circles are congruent then their corresponding chords are congruent.
(if $\overset{\frown}{s} \cong$, chords $\cong$).

The above six theorems are summarized in the following diagram:

$\cong$ central angles $\leftrightarrow$ $\cong$ arcs $\leftrightarrow$ $\cong$ chords.

Problem 4



Given: $\odot O$
$\overline{ES} \parallel \overline{JK}$
$m\widehat{JK} = 78°$
$JE = 1.5$ cm

Find:
a) $\angle s 1-5$
b) $m\widehat{JE}$, $m\widehat{KS}$, KS, $m\widehat{JC}$

Solution

$\angle 1 \cong \angle 4 \cong \angle 5 \cong \angle 3$
(alt. int) (isosc. Δ) (alt. int)

$m\angle 2 = m\widehat{JK} = 78°$

Let $m\angle 1 = x$. Then $x + m\angle 2 + x = 180°$
$2x + 78° = 180°$
$x = 51°$

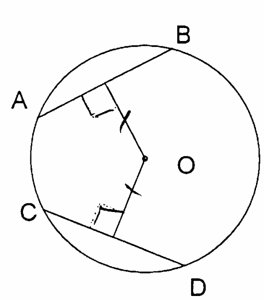
a) Therefore,
$m\angle 1 = 51°$
$m\angle 2 = 78°$
$m\angle 3 = 51°$
$m\angle 4 = 51°$
$m\angle 5 = 51°$

b) $m\widehat{JE} = m\angle 1 = 51°$
$m\widehat{KS} = m\angle 3 = 51°$
$m\widehat{JC} = 180° - 78° = 102°$

$KS = JE = 1.5$ cm
($\cong$ central $\angle$s $\Leftrightarrow$ $\cong$ chords)

Theorem 4 (6.1 – T. 6.1.7 and T. 6.1.8)

Chords are at the same distance from the center of a circle if and only if they are congruent.

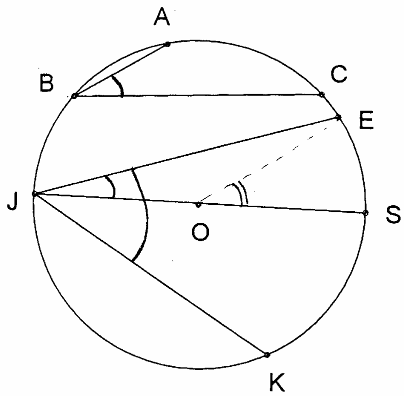


$d(O, \overline{AB}) = d(O, \overline{CD}) \Longleftrightarrow \overline{AB} \cong \overline{CD}$

Note that $d(O, \overline{AB})$ = the distance from O to $\overline{AB}$

Inscribed Angles

There are three different types of inscribed angles when considered in relation to the center of the circle.



1) One side of the angle may contain a diameter, as do $\angle EJS$, $\angle SJK$

2) The circle's center may be in the angle's interior as is the case for $\angle EJK$

3) The center may be in the angle's exterior as it is for $\angle ABC$

Theorem 1
(6.1 – T 6.1.2)

> The measure of an inscribed angle is equal to one-half the degree measure of its intercepted arc. (inscr $\angle = \frac{1}{2}\frown$)

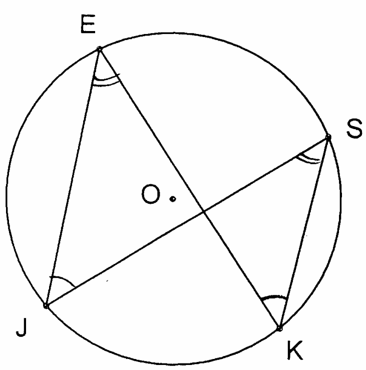
Example: $m\angle ABC = \frac{1}{2}\, m\overset{\frown}{AC}$

$m\angle EJS = \frac{1}{2}\, m\overset{\frown}{ES}$

$m\angle EJK = \frac{1}{2}\, m\overset{\frown}{EK}$

Theorem 2
(6.1 – T 6.1.10)

> If two inscribed angles in a circle intercept the same arc or congruent arcs, then the angles are congruent (inscr $\angle s$ intercept same $\frown$ or $\cong$ $\frown s$ are $\cong$).

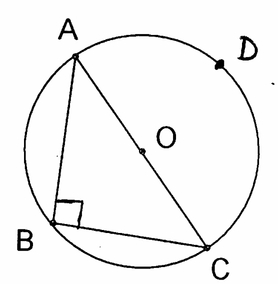


$\angle EJS \cong \angle EKS$

$\angle JEK \cong \angle JSK$

Theorem 3
(6.1 – T 6.1.9)

> If an inscribed angle intercepts a semicircle, then it is a right angle (inscr $\angle$ interc semi $\odot$ is rt $\angle$).

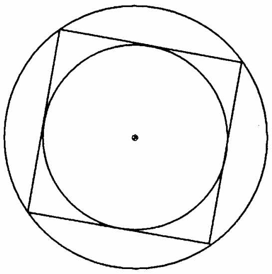


$m\angle ABC = \frac{1}{2}\, m\overset{\frown}{CDA} = \frac{1}{2}\, 180° = 90°$

Polygons inscribed in a circle

Definition Any **polygon is inscribed in a circle** if and only if all its vertices are points of the circle; the **circle is** said to be **circumscribed about the polygon**.

Also, a **circle is inscribed in a polygon** if and only if it is tangent to each of the polygon's sides.

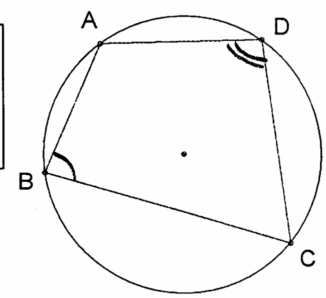


Example:

- the square is inscribed in the larger circle
- the larger circle is circumscribed about the square.
- the smaller circle is inscribed in the square.

Theorem (6.2 – T 6.2.1)

> If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary (if quad inscr in ⊙, opp ∠s supp).



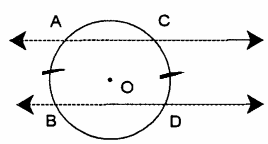
$\angle B$ + $\angle D$ are supplementary because

$m\angle B + m\angle D = \frac{1}{2} m\overset{\frown}{ADC} + \frac{1}{2} m\overset{\frown}{ABC} = \frac{1}{2}(360^\circ) = 180^\circ$

Theorem (6.2 – T 6.2.8)

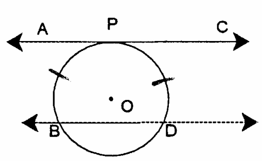
> If two parallel lines intersect a circle, then the arcs of the circle between the parallel lines are congruent (if|| lines intersect ⊙, $\overset{\frown}{s}$ ≅).

a) two secants



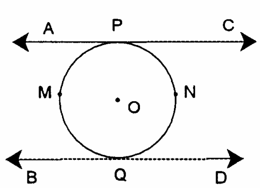
$\overset{\frown}{AB} \cong \overset{\frown}{CD}$

b) secant and tangent



$\overset{\frown}{BP} \cong \overset{\frown}{DP}$

c) two tangents



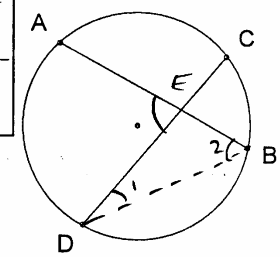
$\overset{\frown}{PMQ} \cong \overset{\frown}{PNQ}$

Chords, Tangents, and Secants

Theorem 1
(6.2 – T 6.2.2)

The measure of an angle formed by two chords that intersect within a circle is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle

(2 chords $\angle = \frac{1}{2}$ sum $\frown$'s).



$< AED$ = ext. ∡ for $\triangle EDB$

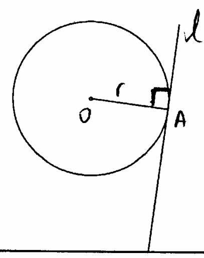
$m < AED = m < 1 + m < 2$

$= \frac{1}{2} m \overset{\frown}{BC} + \frac{1}{2} m \overset{\frown}{AD}$

$\boxed{m < AED = \frac{1}{2}\ m \overset{\frown}{AD} + m \overset{\frown}{BC}}$

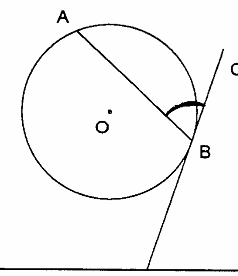
Theorem 2
(6.2 – T 6.2.3)

If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of contact (tan $\perp$ rad to point contact).



Corollary
(6.2 – T 6.2.4)

The measure of an angle formed by a tangent to a circle and a chord drawn to the point of tangency is one-half the measure of its intercepted arc.



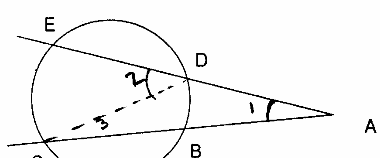
$m < ABC = \frac{1}{2} m \overset{\frown}{AB}$

Theorem 3
(6.2 – T 6.2.5, T 6.2.6, T 6.2.7)

If an angle is formed by

- two secants
  or
- a tangent and a secant
  or
- two tangents

intersecting in the exterior of the circle, then the measure of the angle is one-half the difference of the measures of its intercepted arcs.

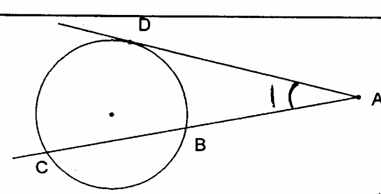


$m<2 = m<1 + m<3$

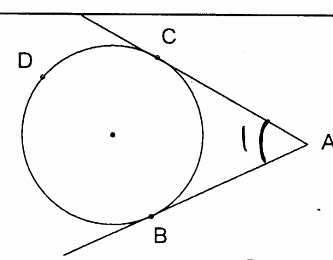
$m<1 = m<2 - m<3$

$= \frac{1}{2} m \overset{\frown}{EC} - \frac{1}{2} m \overset{\frown}{BD}$

$\boxed{m<1 = \frac{1}{2}\left(m \overset{\frown}{EC} - m \overset{\frown}{BD}\right)}$



$m<1 = \frac{1}{2}\left(m \overset{\frown}{DC} - m \overset{\frown}{BD}\right)$

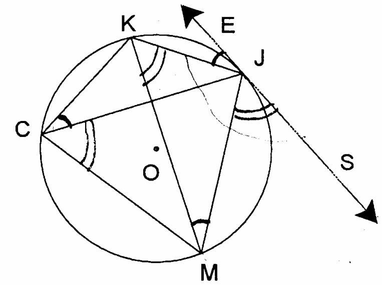


$m<1 = \frac{1}{2}\left(m \overset{\frown}{CDB} - m \overset{\frown}{CB}\right)$

Problem 5 Use the figure to answer the questions.

Given $\odot O$

tan $\overrightarrow{ES}$



a) Name two angles congruent to $\angle KJE$.

< KCJ and < JMK

b) Name two angles congruent to $\angle JCM$.

< MKJ and < SJM

c) Name three angles supplementary to $\angle KJS$.

< KJE , < KCJ , < JMK

d) Name one angle supplementary to $\angle KCM$.

< KJM

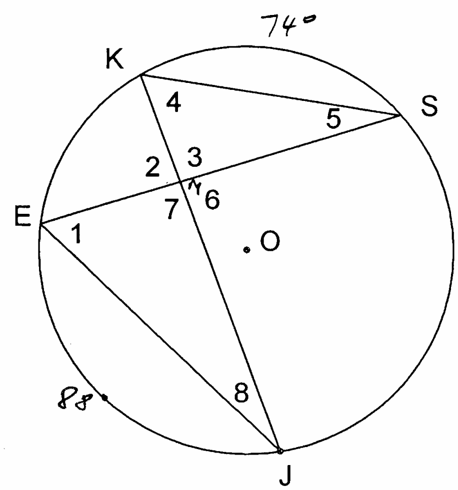
Problem 6 Given $\odot O$

$m\widehat{EJ} = 88^\circ$

$m\widehat{KS} = 74^\circ$

$m\angle 8 = \frac{1}{3} m\angle 2$

Find $m\angle 1-8$



Solution

$m\angle 7 = m\angle 3 = \frac{1}{2}(m\widehat{EJ} + m\widehat{KS})$
(vertical <s) $= \frac{1}{2}(88^\circ + 74^\circ) = 81^\circ$

$m\angle 2 = m\angle 6 = \frac{1}{2}(m\widehat{KE} + m\widehat{SJ})$
(vertical <s)

but $m\widehat{KE} + m\widehat{SJ} = 360^\circ - (m\widehat{EJ} + m\widehat{KS})$
$= 360^\circ - 162^\circ = 198^\circ$

so $m\angle 2 = m\angle 6 = \frac{1}{2}(198^\circ) = 99^\circ$

$m\angle 8 = \frac{1}{3} m\angle 2 = \frac{1}{3}(99^\circ) = 33^\circ$
(given)

$\angle 2$ - ext < $\triangle EMJ$ =>

$m\angle 2 = m\angle 1 + m\angle 8$ =>

$m\angle 1 = 99^\circ - 33^\circ = 66^\circ$

$m\angle 4 = m\angle 1 = 66^\circ$ (inscr <s intercept same ⌒ are ≅)

$m\angle 5 = m\angle 8 = 33^\circ$ (inscr. <s intercept same ⌒ are ≅)

check: $\triangle EMJ$: $m\angle 1 + m\angle 7 + m\angle 8 = 180^\circ$

$m\angle 1 = 66^\circ$
$m\angle 2 = 99^\circ$
$m\angle 3 = 81^\circ$
$m\angle 4 = 66^\circ$
$m\angle 5 = 33^\circ$
$m\angle 6 = 99^\circ$
$m\angle 7 = 81^\circ$
$m\angle 8 = 33^\circ$