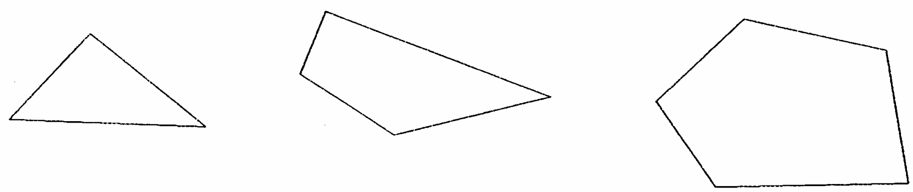
4.1 & 4.2 Properties of a Parallelogram

Definition (2.5) A **polygon** is a closed figure whose sides are line segments that intersect only at endpoints. (*Polygon* is a word of Greek origin that means *many angles*; hence, it implies *many sides*).

Note: 1. We will be working only with **convex polygons**, polygons in which a line segment joining two points in the interior of the polygon has all its points in the interior of the polygon.
2. The angle measures of convex polygons are between 0° and 180°.

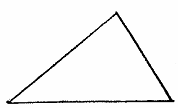
Examples of convex polygons



Definition (2.5) A **diagonal of a polygon** is a line segment that joins two nonconsecutive vertices.

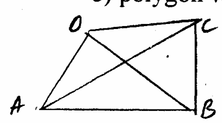
Exercise #1 How many diagonals are in a

a) triangle



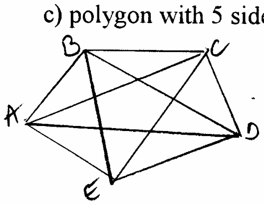
0 diagonals

b) polygon with 4 sides (~~quadrilateral~~)



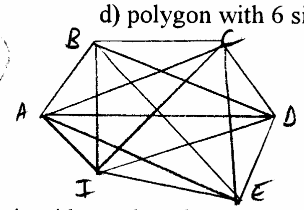
2 diagonals: $\overline{AC}$ and $\overline{BD}$

c) polygon with 5 sides (pentagon)



$\overline{AC}$, $\overline{AD}$ 5 diagonals
$\overline{BD}$, $\overline{BE}$
$\overline{CE}$

d) polygon with 6 sides (hexagon)



9 diagonals

$\overline{AC}$ $\overline{BD}$ $\overline{CE}$ $\overline{DF}$
$\overline{AD}$ $\overline{BE}$ $\overline{CF}$
$\overline{AE}$ $\overline{BF}$

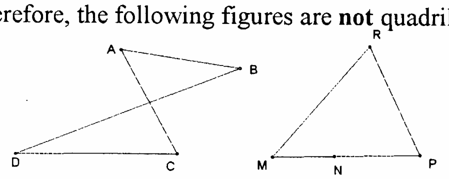
Definition (2.5) A **regular polygon** is a polygon that has its sides and angles congruent.

Definition (4.1) A **quadrilateral** is a polygon that has four sides.

Note: - We will work only with quadrilaterals whose sides are coplanar.
- Special quadrilaterals (squares, rectangles, rhombuses, parallelograms, and trapezoids) occur in various practical circumstances, such as architectural design, construction materials, fabric design, and urban planning.

Important! ABCD is a quadrilateral iff all points are coplanar, no three of which are collinear, and each segment intersects exactly two others, one at each endpoint.

Therefore, the following figures are **not** quadrilaterals:

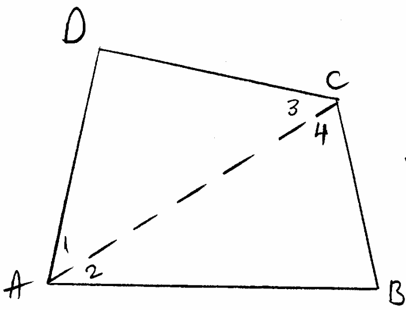


Property (2.5 - C 2.5.4)

The sum of the interior angles of a quadrilateral is 360°.

Given $ABCD$ quad.

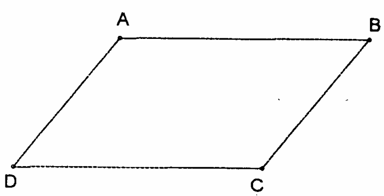
Prove $m\angle A + m\angle B + m\angle C + m\angle D = 360°$



Proof

Statements	Reasons
1. $ABCD$ quad.	1. given
2. Draw $\overline{AC}$	2. 2 points determine a line
3. $m\angle 1 + m\angle 3 + m\angle D = 180°$ ($\triangle ADC$) $m\angle 2 + m\angle 4 + m\angle B = 180°$ ($\triangle ABC$)	3. Sum $\angle$'s in $\triangle = 180°$
4. $(m\angle 1 + m\angle 3 + m\angle D) + (m\angle 2 + m\angle 4 + m\angle B) = 360°$	4. + prop. of equality.
5. $(m\angle 1 + m\angle 2) + (m\angle 3 + m\angle 4) + m\angle B + m\angle D = 360°$	5. Associative prop. of +
6. $m\angle A = m\angle 1 + m\angle 2$ and $m\angle C = m\angle 3 + m\angle 4$	6. Angle-Addition Postulate
7. $m\angle A + m\angle B + m\angle C + m\angle D = 360°$	7. Substitution

Definition (4.1) A **parallelogram** is a quadrilateral whose opposite sides are parallel.



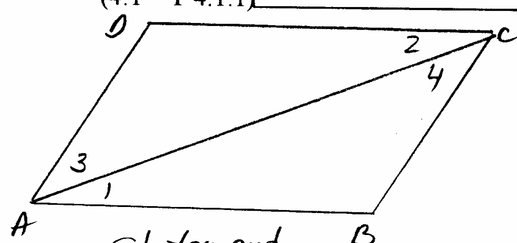
$ABCD$ quadrilateral

$ABCD$ parallelogram iff $\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{BC}$

The defining property for a parallelogram is that it si a quadrilateral whose opposite sides are parallel. Many other properties follow from this. The most significant feature of the figure is that for either pair of opposite sides, the other two sides and the diagonals are transversals. Thus, the theory of parallel lines and transversals may be used to prove properties of parallelograms. This theory and that for congruent triangles provide the needed tools for study of parallelograms.

Theorem 1 (4.1 – T 4.1.1)

A diagonal of a parallelogram separates it into two congruent triangles.



Given: $\square ABCD$ with $\overline{AC}$ diagonal

Prove: $\triangle ACD \cong \triangle CAB$

Proof

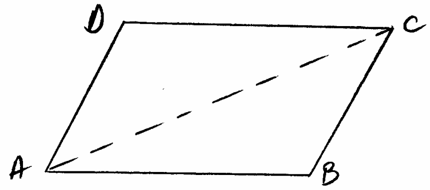
Statements	Reasons.
1. $\square ABCD$	1. given
2. $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$	2. def. of parallelogram (opp. sides $\parallel$)
3. $\angle 1 \cong \angle 2$	3. Alt. int $\angle$'s ($\overline{AB} \parallel \overline{CD}$ and transv. $\overline{AC}$)
4. $\angle 3 \cong \angle 4$	4. Alt. int $\angle$'s ($\overline{AD} \parallel \overline{BC}$ and transv. $\overline{AC}$)
5. $\triangle ACD$, $\triangle CAB$: $\overline{AC} \cong \overline{AC}$; $\angle 3 \cong \angle 4$; $\angle 2 \cong \angle 1$	5. reflexive $\cong$; (4); (3)
6. $\triangle ACD \cong \triangle CAB$	6. ASA

Properties of Parallelograms

Corollaries
(4.1 – C 4.1.2,3,4,5)

1. The opposite sides of a parallelogram are congruent (opp sides ▱ $\cong$).
2. The opposite angles of a parallelogram are congruent (opp $\angle$'s ▱ $\cong$).
3. Any two consecutive angles of a parallelogram are supplementary (consec $\angle$'s ▱ supp).
4. The diagonals of a parallelogram bisect each other (diags ▱ bisect each other).

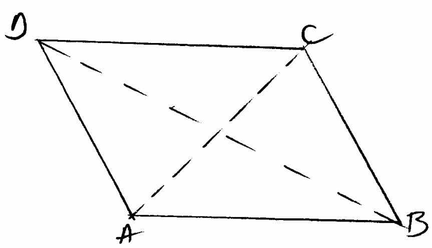
① Given ▱ABCD
Prove $\overline{AB} \cong \overline{CD}$
$\overline{AD} \cong \overline{BC}$



Proof of ①

Statements	Reasons
1. ▱ ABCD	1. given
2. Draw $\overline{AC}$	2. 2 point determine a line
3. $\triangle ACD \cong \triangle CAB$	3. ▱, diag. forms $\cong \triangle$'s
4. $\overline{AD} \cong \overline{BC}$ $\overline{DC} \cong \overline{AB}$	4. CPCTC

② Given ▱ABCD
Prove $\angle A \cong \angle C$
$\angle B \cong \angle D$

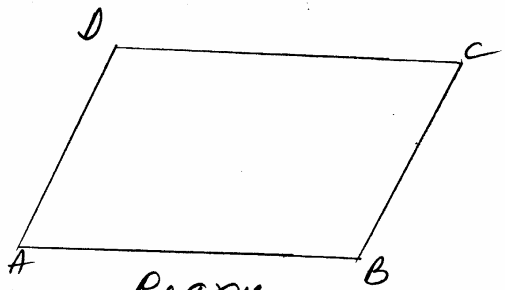


Proof of ②

Statements	Reasons
1. ▱ ABCD	1. given
2. Draw $\overline{AC}$ and $\overline{BD}$	2. 2 point determine a line
3. $\triangle ACD \cong \triangle CAB$	3. in ▱, diag. form $\cong \triangle$'s
4. $\angle B \cong \angle D$	4. CPCTC
5. $\triangle ABD \cong CDB$	5. in ▱, diag. form $\cong \triangle$'s
6. $\angle A \cong \angle C$	6. CPCTC

(3) Given: ▱ ABCD

Prove: ∠A supp. ∠B
∠B supp. ∠C
∠C supp. ∠D
∠D supp. ∠A

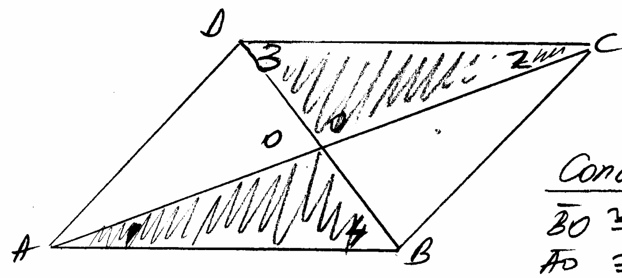


Proof of (3)

Statements	Reasons
1. ▱ ABCD	1. given
2. $\overline{AB} \parallel \overline{CD}$	2. def. of ▱ (▱ iff. opp. sides ∥)
3. ∠A supp. ∠D ∠B supp. ∠C	3. same side int. ∠'s are supplementary $\overline{AB} \parallel \overline{CD}$ and transv. $\overline{AD}$ $\overline{AB} \parallel \overline{CD}$ and transv. $\overline{BC}$
4. $\overline{AD} \parallel \overline{BC}$	4. def. of ▱ (▱ iff opp. sides ∥)
5. ∠A supp. ∠B ∠D supp. ∠C	5. same side int. ∠'s are supplementary $\overline{AD} \parallel \overline{BC}$ and transv. $\overline{AB}$ $\overline{AD} \parallel \overline{BC}$ and transv. $\overline{DC}$

(4) Given: ▱ ABCD
$\overline{AC}$, $\overline{BD}$ diagonals

Prove: $\overline{AC}$ bisects $\overline{BD}$
$\overline{BD}$ bisects $\overline{AC}$



Conditions!
$\overline{BO} \cong \overline{DO}$
$\overline{AO} \cong \overline{CO}$

Proof of (4)

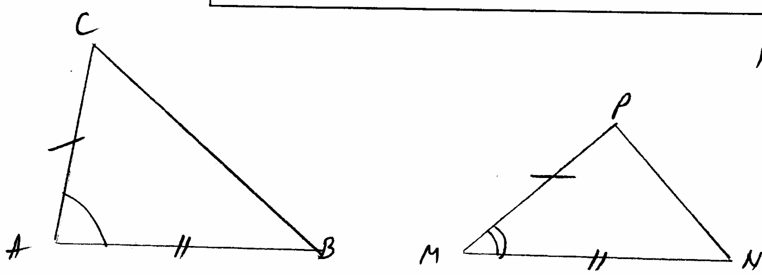
Statements	Reasons
1. ▱ ABCD, diag. $\overline{AC}$, $\overline{BD}$	1. given
2. $\overline{AB} \parallel \overline{DC}$	2. def. of ▱ (▱ iff opp. sides ∥)
3. ∠1 ≅ ∠2 ∠4 ≅ ∠3	3. alt. int. ∠'s $\overline{AB} \parallel \overline{DC}$ and transv. $\overline{AC}$ $\overline{AB} \parallel \overline{DC}$ and transv. $\overline{BD}$
4. $\overline{AB} \cong \overline{CD}$	4. opp. sides ▱ ≅
5. △AOB, △COD { $\overline{AB} \cong \overline{CD}$, ∠1 ≅ ∠2, ∠4 ≅ ∠3	5. { (4), (3), (3)
6. △AOB ≅ △COD	6. ASA
7. $\overline{AO} \cong \overline{CO}$	7. CPCTC
8. $\overline{BD}$ bisects $\overline{AC}$	8. def. of bisector of a segment
9. $\overline{BO} \cong \overline{DO}$	9. CPCTC
10. $\overline{AC}$ bisects $\overline{BD}$	10. same as (8)

Definition The **distance between two parallel lines** is the length of any perpendicular line segment joining the lines.

Definition (4.1) An **altitude of a parallelogram** is a line segment from one vertex that is perpendicular to the opposite side (or to an extension of that side).

Lemma (4.1 – L 4.1.6)

> If two sides of one triangle are congruent to two sides of a second triangle and the included angle of the first triangle is greater than the included angle of the second, then the length of the side opposite the included angle of the first triangle is greater than the length of the side opposite the included angle of the second.



if $\overline{AB} \cong \overline{MN}$

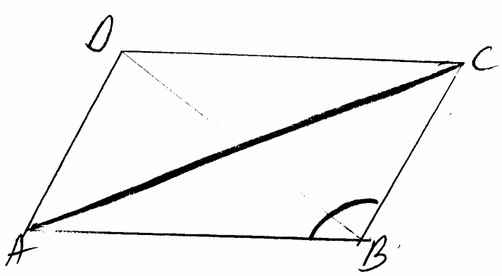
$\overline{AC} \cong \overline{MP}$

$m\angle A > m\angle M$

then $BC > PN$

Theorem 2 (4.1 – T 4.1.7)

> In a parallelogram with unequal pairs of consecutive angles, the longer diagonal lies opposite the obtuse angle.

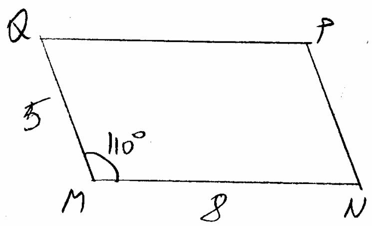


if $\angle B$ - obtuse $(m\angle B > m\angle A)$

$\angle A$ - acute

then $\overline{AC} > \overline{BD}$

Problem #1 (4.1 - #3) MNPQ is a parallelogram. Suppose that MQ =5, MN = 8, and $m\angle M = 110°$. Find: a) QP; b) NP; c) $m\angle Q$; d) $m\angle P$.



MNPQ - parallelogram

$\overline{MN} \cong \overline{PQ}$ (opp sides $\square$ $\cong$)

$\overline{MQ} \cong \overline{NP}$

$\Rightarrow PQ = 8$ and $NP = 5$

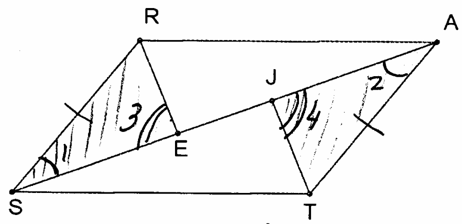
$m\angle M + m\angle Q = 180°$ (consec. $\angle$'s $\square$ supp)

$110° + m\angle Q = 180° \Rightarrow m\angle Q = 70°$

$\angle M \cong \angle P$ (opp $\angle$'s $\square$ $\cong$) $\Rightarrow m\angle P = 110°$

$\angle Q \cong \angle N$ $\quad m\angle N = 70°$

Problem #2



Given: $\square$ *STAR*
$\overline{ER} \parallel \overline{JT}$

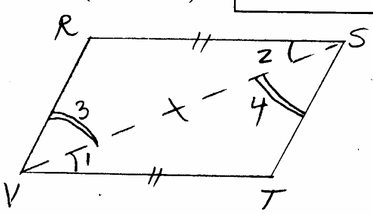
Prove: $\overline{ER} \cong \overline{JT}$

(Condition $\triangle SER \cong \triangle AJT$)

Statements	Reasons.
1. $\square$ STAR	1. given
2. $\overline{SR} \cong \overline{AT}$	2. opp. sides $\square$ $\cong$
3. $\overline{SR} \parallel \overline{AT}$	3. def. of $\square$ ($\square$ iff opp. sides $\parallel$)
4. $\angle 1 \cong \angle 2$	4. alt. int. $\angle$'s ($\overline{SR} \parallel \overline{AT}$ and transv. $\overline{SA}$)
5. $\overline{ER} \parallel \overline{JT}$	5. given
6. $\angle 3 \cong \angle 4$	6. alt. ext. $\angle$'s ($\overline{ER} \parallel \overline{JT}$ and transv $\overline{SA}$)
7. $\triangle SER$, $\triangle AJT$ $\begin{cases} \overline{SR} \cong \overline{AT} \\ \angle 1 \cong \angle 2 \\ \angle 3 \cong \angle 4 \end{cases}$	7. $\begin{cases} (2) \\ (4) \\ (6) \end{cases}$
8. $\triangle SER \cong \triangle AJT$	8. AAS
9. $\overline{ER} \cong \overline{JT}$	9. CPCTC

When is a quadrilateral a parallelogram?

Theorem 3 (4.2 – T 4.2.1): If two sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.



Given: VTSR quadrilateral
$\overline{RS} \parallel \overline{VT}$
$\overline{RS} \cong \overline{VT}$

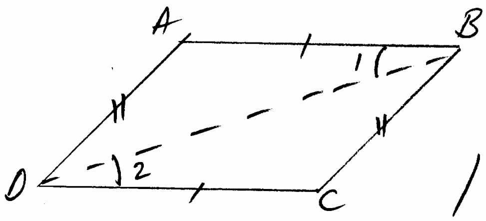
Prove: VTSR parallelogram

(Condition $\overline{VR} \parallel \overline{TS}$)

Statements Proof	Reasons
1. $\overline{RS} \parallel \overline{VT}$; $\overline{RS} \cong \overline{VT}$	1. given
2. Draw $\overline{VS}$	2. 2 points determine a line
3. $\angle 1 \cong \angle 2$	3. alt. int. $\angle$'s ($\overline{RS} \parallel \overline{VT}$ and transv. $\overline{VS}$)
4. $\triangle RSV$, $\triangle TVS$ $\begin{cases} \overline{VS} \cong \overline{VS} \\ \angle 2 \cong \angle 1 \\ \overline{RS} \cong \overline{VT} \end{cases}$	4. $\begin{cases} \text{reflexive} \cong \\ (3) \\ \text{given} \end{cases}$
5. $\triangle RSV \cong \triangle TVS$	5. SAS
6. $\angle 3 \cong \angle 4$	6. CPCTC
7. $\overline{RV} \parallel \overline{ST}$	7. $\parallel$ iff alt. int $\angle$'s $\cong$ ($\overline{RV}$ and $\overline{ST}$ with transv. $\overline{VS}$)
8. VTSR is a parallelogram	8. def. of $\square$ ($\square$ iff. opp. sides $\parallel$)

Theorem 4 (4.2 – T 4.2.2)

If both pairs of opposite sides of a quadrilateral are congruent, then it is a parallelogram.

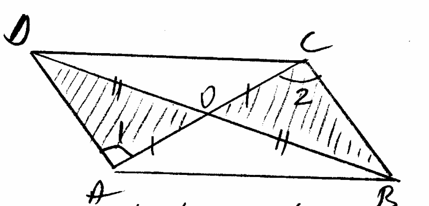


Given: ABCD quadrilateral
$\overline{AB} \cong \overline{DC}$
$\overline{AD} \cong \overline{BC}$
Prove: ABCD is a parallelogram
(Condition: $\overline{AB} \parallel \overline{DC}$)

Statements	Reasons
1. ABCD - quadrilateral	1. given
2. Draw $\overline{BD}$	2. 2 points determine a line
3. $\triangle ABD$, $\triangle CDB$: $\overline{BD} \cong \overline{BD}$; $\overline{AB} \cong \overline{DC}$; $\overline{AD} \cong \overline{BC}$	3. reflexive $\cong$; given; given
4. $\triangle ABD \cong \triangle CDB$	4. SSS
5. $\angle 1 \cong \angle 2$	5. CPCTC
6. $\overline{AB} \parallel \overline{DC}$	6. $\parallel$ iff alt. int $\angle$'s $\cong$ ($\overline{AB}$ and $\overline{DC}$ with transv. $\overline{BD}$)
7. ABCD = parallelogram	7. opp. sides $\parallel$ and $\cong$ ($\overline{AB} \parallel \cong \overline{DC}$)

Theorem 5 (4.2 – T 4.2.3)

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

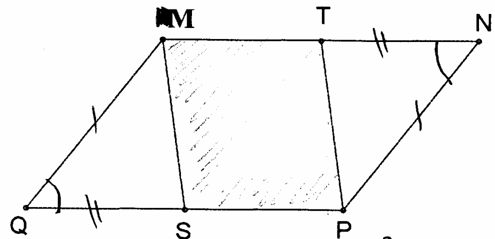


Given: ABCD - quadrilateral
$\overline{AC}$ and $\overline{BD}$ - diagonals
$\overline{AC}$ bisects $\overline{BD}$
$\overline{BD}$ bisects $\overline{AC}$
Prove: ABCD = parallelogram
(Condition: $\overline{AD} \parallel \cong \overline{BC}$)

Statements	Reasons
1. ABCD - quad, $\overline{AC}$, $\overline{BD}$ diag.	1. given
2. $\overline{AC}$ bisects $\overline{BD}$	2. given
3. $\overline{BO} \cong \overline{DO}$	3. def. of bisector of a segment
4. $\overline{BD}$ bisects $\overline{AC}$	4. given
5. $\overline{AO} \cong \overline{CO}$	5. same as (3)
6. $\triangle AOD$, $\triangle COB$: $\overline{AO} \cong \overline{CO}$; $\overline{BO} \cong \overline{DO}$; $\angle AOD \cong \angle BOC$	6. (5); (3); vertical angles
7. $\triangle AOD \cong \triangle COB$	7. SAS
8. $\overline{AD} \cong \overline{BC}$ and $\angle 1 \cong \angle 2$	8. CPCTC
9. $\overline{AD} \parallel \overline{BC}$	9. $\parallel$ iff alt. int $\angle$'s $\cong$ ($\overline{AD}$ and $\overline{BC}$ with transv. $\overline{AC}$)
10. ABCD ▱	10. opp. sides $\parallel$ and $\cong$

Problem #3

(4.2 - #22)



Given: $\square MNPQ$

T midpoint of $\overline{MN}$

S midpoint of $\overline{QP}$

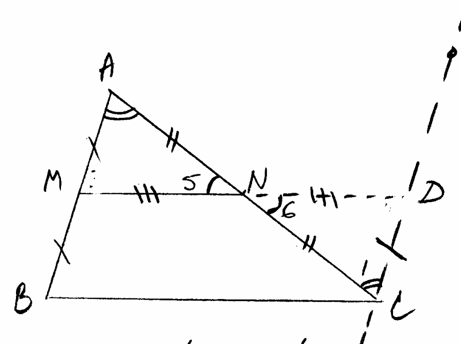
Prove: $\triangle QMS \cong \triangle NPT$

$MSPT$ is a parallelogram

Statements	Reasons
1. $MNPQ$ $\square$	1. given
2. $\overline{MQ} \cong \overline{NP}$, $\overline{MN} \cong \overline{QP}$	2. opp sides $\square$ $\cong$
3. $\angle Q \cong \angle N$	3. opp $\angle$'s $\square$ $\cong$
4. T- midpoint of $\overline{MN}$ S - midpoint of $\overline{QP}$	4. given
5. $TN = \frac{1}{2} MN = MT$ $QS = \frac{1}{2} QP = SP$	5. midpoint divides segment into 2 equal parts
6. $MN = QP$	6. def. of congruent segments
7. $TN = QS$ and $MT = SP$ (5,6)	7. substitution/transitivity
8. $\overline{TN} \cong \overline{QS}$ and $\overline{MT} \cong \overline{SP}$	8. def. of $\cong$ segments
9. $\triangle QMS$, $\triangle NPT$ $\left\{ \begin{array}{l} \overline{MQ} \cong \overline{PN} \\ \overline{QS} \cong \overline{NT} \\ \angle Q \cong \angle N \end{array} \right.$	9. $\left\{ \begin{array}{l} (2) \\ (8) \\ (3) \end{array} \right.$
10. $\triangle QMS \cong \triangle NPT$	10. SAS
11. $\overline{MS} \cong \overline{PT}$	11. CPCTC
12. $MSPT$ $\left\{ \begin{array}{l} \overline{MS} \cong \overline{PT} \\ \overline{MT} \cong \overline{SP} \end{array} \right.$	12. $\left\{ \begin{array}{l} (11) \\ (8) \end{array} \right.$
13. $MSPT$ = parallelogram	13. $\square$ if both opp. sides $\cong$

Theorem 6
(4.2 – T 4.2.5)

The segment that joins the midpoints of two sides of a triangle is parallel to the third side and has a length equal to one-half the length of the third side.



Given: $\triangle ABC$
M = midpoint of $\overline{AB}$
N = midpoint of $\overline{AC}$

Prove $\overline{MN} \parallel \overline{BC}$
$MN = \frac{1}{2} BC$

Statements	Reasons
1. $\triangle ABC$	1. given
2. M- midpoint of $\overline{AB}$ N - midpoint of $\overline{AC}$	2. given
3. $\overline{AM} \cong \overline{MB}$ and $\overline{AN} \cong \overline{NC}$	3. def. of midpoint.
4. construct line $\overleftrightarrow{CE} \parallel \overline{AB}$	4. Through a point not on the line, there is only one line parallel to the given line
5. extend $\overline{MN}$ to meet $\overleftrightarrow{CE}$ at D	5. 2 points determine a line ($\overleftrightarrow{MN}$)
6. $\angle A \cong \angle 1$	6. alt. int ∡'s ($\overline{AB} \parallel \overleftrightarrow{CE}$, transv. $\overline{AC}$)
7. $\triangle ANM$ $\triangle CND$ { $\overline{AN} \cong \overline{NC}$ $\angle A \cong \angle 1$ $\angle 5 \cong \angle 6$	7. (3) (6) vertical angles
8. $\triangle ANM \cong \triangle CND$	8. ASA
9. $\overline{AM} \cong \overline{DC}$ $\overline{MN} \cong \overline{ND}$	9. CPCTC
10. $\overline{BM} \cong \overline{DC}$	10. transitivity $\cong$
(3,9) 11. BMDC = parallelogram	11. opp sides ∥ and $\cong$ ($\overline{BM} \overset{\parallel}{\cong} \overline{DC}$)
12. $\overline{MN} \parallel \overline{BC}$	12. def. of ▱ (▱ iff. opp sides ∥)
13. $\overline{MD} \cong \overline{BC}$	13. opp. sides ▱ $\cong$.
14 $MN + ND = MD$	14. Segment - Addition Postulate
15 $MN = ND$	15. Def. of $\cong$ segments
(9) 16. $MN + MN = MD$ (14,15) $2MN = MD$	16. Substitution

17. $MN = \frac{1}{2} MO$	17. Division prop. of equality
18. (13) $MO = BC$	18. Def. of $\cong$ segments
19. (17, 18) $MN = \frac{1}{2} BC$	