# 3.5 Inequalities in a Triangle

There are several important order relations involving the measures of the sides and angles of a triangle.

**Definition** If $a, b \in \mathbb{R}$, then $a > b$ if and only if there is $p > 0$ such that $a = b + p$.

**Properties of Inequalities**
(A 3)

Addition: If $a < b$, then $a + c < b + c$ for any $c$.

Subtraction: If $a < b$, then $a - c < b - c$ for any $c$.

Multiplication: If $a < b$, then $a \cdot c < b \cdot c$ for any $c > 0$.
$a \cdot c > b \cdot c$ for any $c < 0$.

Division: If $a < b$, then $\frac{a}{c} < \frac{b}{c}$ for any $c > 0$.
$\frac{a}{c} > \frac{b}{c}$ for any $c < 0$.

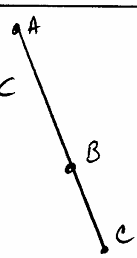
The Transitive Property of Inequality (Order)
If $a, b, c \in \mathbb{R}$ and if $a < b$ and $b < c$, then $a < c$

**Lemma 1**
(3.5 – L 3.5.1)

The measure of a line segment is greater than the measure of any of its parts.

Given: $\overline{AC}$
$B \in \overline{AC}$, $A$-$B$-$C$

Prove: $AC > AB$
$AC > BC$




Proof

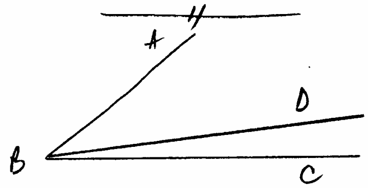
| Statements | Reasons |
|---|---|
| 1. $B \in \overline{AC}$, $A-B-C$ | 1. Given |
| 2. $AB + BC = AC$ | 2. Segment-Addition Postulate |
| 3. $BC > 0$ | 3. Ruler Postulate |
| 4. $AC > BC$ | 4. definition of $a > b$ |
| Similarly, $AC > AB$. | |

**Lemma 2**
(3.5 – L 3.5.2)

The measure of an angle is greater than the measure of any of its parts.

Given: $\angle ABC$
$\overrightarrow{BD}$ in int $\angle ABC$

Prove: $m\angle ABC > m\angle ABD$
$m\angle ABC > m\angle CBD$




Proof

| Statements | Reasons |
|---|---|
| 1. $D \in$ int $\angle ABC$ | 1. Given |
| 2. $m\angle ABD + m\angle DBC = m\angle ABC$ | 2. Angle-Addition Postulate |
| 3. $m\angle DBC > 0$ | 3. Protractor Postulate |
| 4. $m\angle ABC > m\angle ABD$ | 4. definition of $a > b$ |
| Similarly, $m\angle ABC > m\angle ABD$. | |

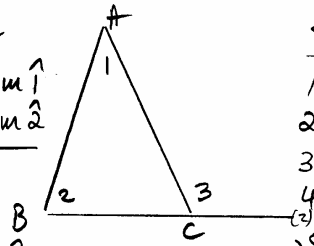
**Lemma 3** (3.5 – L 3.5.3)

The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.

Given: $\triangle ABC$

Prove: $m\hat{3} > m\hat{1}$

$m\hat{3} > m\hat{2}$



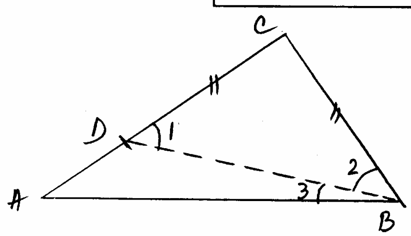

Proof

| Statement | Reasons |
|---|---|
| 1. $\angle 3$ exterior $\triangle ABC$ | 1. given |
| 2. $m\angle 3 = m\angle 1 + m\angle 2$ | 2. ext $\angle$ = sum nonadj int. $\angle$'s |
| 3. $m\angle 2 > 0$ | 3. Protractor Postulate |
| 4. $m\angle 3 > m\angle 1$ (2) | 4. Definition of $a>b$ |
| 5. $m\angle 1 > 0$ | 5. Protractor Postulate |
| 6. $m\angle 3 > m\angle 2$ (2) | 6. Def of $a>b$ |

Note that $\angle 3$ and $\hat{3}$ means $\angle 3$ (angle 3)

**Theorem 1** (3.5 – T 3.5.6)

If the lengths of two sides of a triangle are unequal, then the measures of the angles opposite them are unequal and the larger angle is opposite the longer side.
( 2 sides $\Delta \neq$, opp $\angle's \neq$ same order)




Given: $\triangle ABC$

$CA > CB$

Prove: $m\angle ABC > m\angle A$

Proof

| Statement | Reasons |
|---|---|
| 1. $\triangle ABC$ with $CA > CB$ | 1. given. |
| 2. construct $\overline{CD} \cong \overline{CB}$, $C-D-A$ | 2. can copy segment |
| 3. Draw $\overline{DB}$ | 3. 2 points determine a line |
| 4. $\angle 1 \cong \angle 2$ | 4. $\triangle DCB$, if 2 sides $\cong$, opp. $\angle$'s $\cong$ |
| 5. $m\angle 1 = m\angle 2$ | 5. definition of congruent angles. |
| 6. $m\angle ABC = m\angle 3 + m\angle 2$ | 6. Angle-Addition Postulate |
| 7. $m\angle ABC = m\angle 3 + m\angle 1$ | 7. Substitution |
| 8. $m\angle 3 > 0$ | 8. Protractor Postulate |
| 9. $m\angle ABC > m\angle 1$ (7,8) | 9. definition of $a>b$ |
| 10. $\angle 1$ = exterior $\angle$ of $\triangle ADB$ | 10. By construction |
| 11. $m\angle 1 > m\angle A$ | 11. ext. $\angle$ of $\triangle$ > nonadj int. $\angle$ |
| 12. $m\angle ABC > m\angle A$ (9,11) | 12. Transitivity |

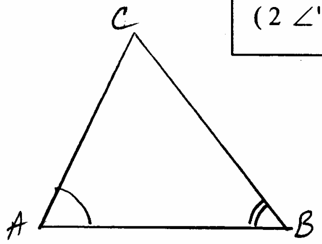
q.e.d

Note: The relationship described in the above theorem extends to all sides and all angles of a triangle. That is, the largest of the three angles of a triangle is opposite the longest side, and the smallest angle is opposite the shortest side.

The converse of Theorem 1 is also true.

**Theorem 2**
(3.5 – T 3.5.7)

> If the measures of two angles of a triangle are unequal, then the lengths of the sides opposite them are unequal and the longer side is opposite the larger angle.
> ( 2 $\angle$'s $\Delta \neq$, opp sides $\neq$ same order)




Given $\triangle ABC$
$m\measuredangle A > m\measuredangle B$ (*)

Prove $BC > AC$

Proof (Indirect)

Assume $BC \not> AC$
Then: case (1) $BC < AC$
OR
case (2) $BC = AC$

- Case (1) $\boxed{BC < AC}$
$\Rightarrow m\measuredangle A < m\measuredangle B$ (Th: 2 sides $\Delta \neq$, opp $\measuredangle$'s $\neq$ same order)
Contradiction with given (*)
$\Rightarrow$ case (1) is not possible.
- Case (2) $\boxed{BC = AC}$
$\Rightarrow m\measuredangle A = m\measuredangle B$ (Th: 2 sides $\Delta =$, opp $\measuredangle$'s $=$)
Contradiction with given (*)
$\Rightarrow$ case (2) is not possible

Therefore, our assumption is false $\Rightarrow BC > AC$

**Corollary 1**
(3.5 – C 3.5.8)

> The perpendicular segment from a point to a line is the shortest segment that can be drawn from the point to the line.

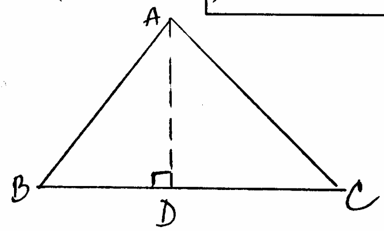

see textbook page 148
Corollary 3.5.8

**Corollary 2**
(3.5 – C 3.5.9)

> The perpendicular segment from a point to a plane is the shortest segment that can be drawn from the point to the plane.


see textbook page 148
Corollary 3.5.9.

**Theorem 3** **The Triangle Inequality**
(3.5 – T 3.5.10) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.




Given: $\triangle ABC$

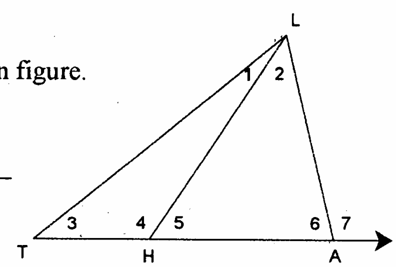
Prove: $BA + CA > BC$
$BA + BC > AC$
$CA + BC > AB$

Proof

| Statements | Reasons |
|---|---|
| 1. $\triangle ABC$ | 1. given |
| 2. Draw $\overline{AD} \perp \overline{BC}$ | 2. From a point not on a line, there is exactly one line $\perp$ to the given line. |
| 3. $BA > BD$<br>$CA > DC$ | 3. The $\perp$ segment = shortest segment (Corollary 1 above) |
| 4. $BA + CA > BD + DC$ | 4. Addition Property of inequality |
| 5. $D \in \overline{BC}$, $B-D-C$ | 5. given |
| 6. $BD + DC = BC$ | 6. Segment-Addition Postulate |
| 7. $BA + CA > BC$ | 7. Substitution (4,6) |

Similarly, $BA + BC > AC$ and $CA + BC > AB$.

Problem #1 Write the theorem that justifies each statement. Refer to the given figure.



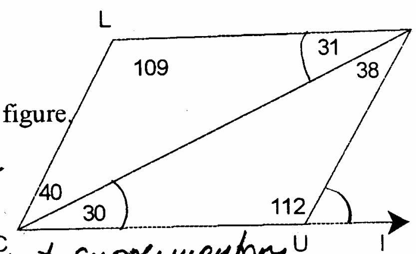

a) $m\measuredangle 7 > m\measuredangle 5$ ext $\measuredangle$ > nonadj. int $\measuredangle$

b) $m\measuredangle 4 > m\measuredangle 6$ ext $\measuredangle$ > nonadj int $\measuredangle$

c) If $LH < TL$, then $m\measuredangle 3 < m\measuredangle 4$ $\triangle LHT$, if 2 sides $\neq$, opp $\measuredangle$'s $\neq$ same order

d) If $m\measuredangle 6 > m\measuredangle 3$, then $TL > LA$. $\triangle LTA$, if 2 $\measuredangle$'s $\neq$, opp. sides $\neq$ same order

e) $LA + TA > TL$ sum 2 sides $\triangle$ > 3rd side




Problem #2 Write the theorem that justifies each statement. Refer to the given figure.

a) $\overline{LE} \nparallel \overline{CU}$ lines cut by trans, alt int ∡'s not ≅ (LE and CU, CE trans, 31° ≠ 30°)

b) $\overline{LC} \nparallel \overline{EU}$ lines cut by trans, same side int ∡'s not supplementary (LC and UE, CU trans, 40°+30°+112° ≠ 180°)

c) $LE > LC$ △LEC, if 2∡'s ≠, opp sides △ ≠ same order

d) $EU + UC > EC$ △CUE; sum 2 sides > 3rd side

e) $LE < LC + EC$ △LCE; sum 2 sides > 3rd side

f) $EU < UC$ △EUC; if 2∡'s ≠, opp sides ≠ same order

g) $m\angle EUI > m\angle ECU$ △ECU; ext∡ > nonadj. int ∡

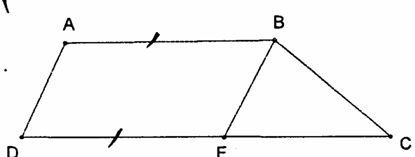
Problem #3 (3.5 - #13) Is it possible to draw a triangle whose sides measure:

a) 8, 9, and 10 in? yes iff $\begin{cases} 8 < 9+10 & \text{true} \\ 9 < 8+10 & \text{true} \\ 10 < 8+9 & \text{true} \end{cases}$ Therefore, yes.

b) 8, 9, and 17 m? yes iff $\begin{cases} 8 < 9+17 & \text{true} \\ 9 < 8+17 & \text{true} \\ 17 < 8+9 & \text{false} \end{cases}$ Therefore, NO.

c) 8, 9, and 18 ft?

yes iff $\begin{cases} 8 < 9+18 & \text{True} \\ 9 < 8+18 & \text{True} \\ 18 < 8+9 & \text{False} \end{cases}$ Therefore, NO.




Problem #4 (3.5 - #24) Given a quadrilateral ABCD with $\overline{AB} \cong \overline{DE}$, show that $DC > AB$.

| Statements | Reasons |
|---|---|
| 1. ABCD, $\overline{AB} \cong \overline{DE}$ | 1. given |
| 2. AB = DE | 2. Definition of congruent segments |
| 3. DC = DE + EC | 3. Segment-Addition Postulate |
| 4. DC = AB + EC (2,3) | 4. Substitution |
| 5. EC > 0 | 5. Ruler Postulate |
| 6. DC > AB (4, 5) | 6. Definition of a > b |

Problem #5 (3.5 - #27, #29) a) The sides of a triangle have lengths of 4, 6, and x. Write an inequality that states the possible values of x.

The length of any side of a △ must lie between the sum and difference of the lengths of the other two sides (Theorem 3.5.10)

$6-4 < x < 6+4$

$\boxed{2 < x < 10}$

b) If the lengths of two sides of a triangle are represented by 2x+5 and 3x+7 ( in which x is positive), describe in terms of x the possible lengths of the third side whose length is represented by y.

$(3x+7)-(2x+5) < y < (3x+7)+(2x+5)$

$3x+7-2x-5 < y < 3x+7+2x+5$

$\boxed{x+2 < y < 5x+12}$