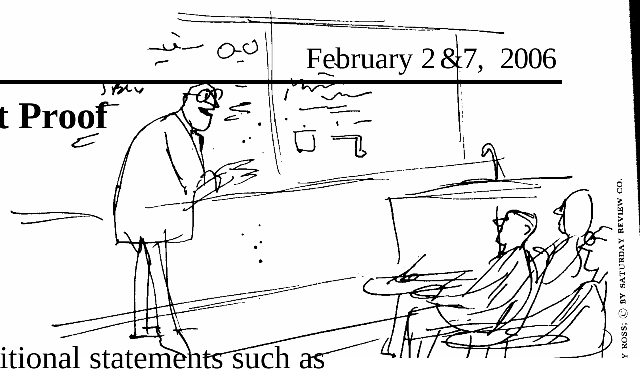
Section 2.2 Indirect Proof



A series of lessons in a subject that contradicted each other would make that subject very confusing. Yet, in reasoning deductively in geometry, it is sometimes helpful to try to arrive at contradictions deliberately!

Proofs → Direct - consist of a sequence of conditional statements such as $a \to b,\ b \to c,\ c \to d,\ d \to e$ that "link together" to yield a theorem, which in this case is $a \to e$.

Proofs → Indirect - involve reasoning to a contradiction.

Consider the following puzzle:

Emerson, Lake, and Palmer are different heights. Who is the tallest and who is the shortest if *only one* of the following statements is true?

1. Emerson is the tallest.
2. Lake is not the tallest.
3. Palmer is not the shortest.

One way of starting to figure this puzzle out would be to assume that the first statement is the one that is true, which would mean that the other two are false. This results in the following three statements:

1. Emerson is the tallest.
2. ______________________________
3. ______________________________

What is wrong? ____

Therefore, it is reasonable to conclude that, if there is nothing wrong with the puzzle itself, our assumption that the first statement is true must be wrong. Therefore, it is either the second or the third statement that is true.

On the assumption that the second statement is the true one, we get:

1. ______________________________
2. ______________________________
3. ______________________________

What is the contradiction? ____

So, our assumption that the second statement is true is wrong.

That leaves just one more possibility if the puzzle has a solution at all; namely, that the third statement is the true one. We have:

1. ______________________________
2. ______________________________
3. ______________________________

This time there is no contradiction. Evidently Lake is the tallest of the three and Emerson is the shortest.

The basic pattern in proving a theorem, say $P \to Q$, indirectly is to begin by assuming *not* Q ($\sim Q$). It is by reasoning from *this* statement that we hope to arrive at a contradiction. The statements Q and $\sim Q$ are called opposites of each other.

Exercise #1 — Write the opposites (negation) of the following statements:

a) Seven is a prime number. ______________________________

b) Mr. Spock does not like contradictions. ______________________________

c) A line contains at least two points. ______________________________

To prove a theorem indirectly, we begin by assuming the opposite of its conclusion.

Exercise #2 — Write the opposite of the conclusion of each of the following theorems.

a) If a number is odd, its square is odd. ______________________________

b) If two lines intersect, they intersect in no more than one point.

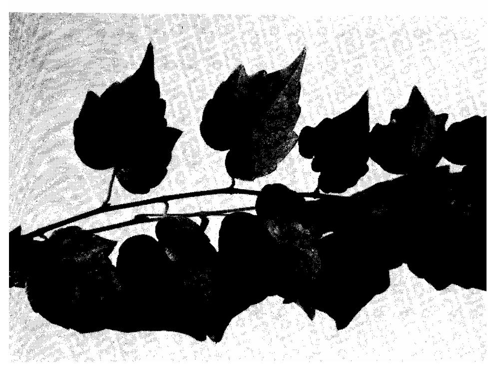
____

c) In a plane, two lines perpendicular to a third line are parallel to each other.

____

Example of an indirect proof.

It is sufficient to know that poison ivy has leaves in groups of three to prove that the plant in this photograph is not poison ivy.



a) With what assumption would we begin the proof?

____

b) What conclusion follows from this assumption?

____

c) What does this conclusion contradict?

____

Since our initial assumption led to a contradiction, it must be false. In other words, the statement “The plant in this photograph is poison ivy” is false.

d) What statement, then, must be true? ______________________________

<u>Method of Indirect Proof</u>

To prove the statement $P \to Q$ or to complete the proof problem of the form

Given: P

Prove: Q

where Q may be a negation, use the following steps:

1. Suppose that $\sim Q$ is true.
2. Reason from the supposition until you reach a contradiction.
3. Note that the supposition claiming that $\sim Q$ is true must be false and that Q must therefore be true.

Step 3 completes the proof.

Note that the contradiction that is discovered in an indirect proof often has the form $\sim P$. Thus the assumed statement $\sim Q$ has forced the conclusion $\sim P$, asserting that $\sim Q \to \sim P$ is true. Then the desired theorem $P \to Q$ (the contrapositive of $\sim Q \to \sim P$) is also true.

<u>Exercise #3</u> (2.2: #20)

If a and b are positive numbers, then $\sqrt{a^2 + b^2} \neq a + b$.

Given:

Prove:

Proof:

<u>Exercise #4</u> (2.2: #21)

The midpoint of a line segment is unique.

Given:

Prove:

Proof:

<u>Exercise #5</u>
(2.2 Example 4)

If two parallel planes are intersected by a third plane, then the lines of intersection are parallel.

Given:

Prove:

Proof:

<u>Exercise #6</u>
(2.2 Example 5)

The angle bisector of an angle is unique.

Given:

Prove:

Proof:

2.3 More on Parallel Lines

Theorem

If two lines are each parallel to a third line, then these lines are parallel to each other.

Hypothesis:

Conclusion:

Proof:

Theorem

If two coplanar lines are each perpendicular to a third line, then these lines are parallel to each other.

Hypothesis:

Conclusion:

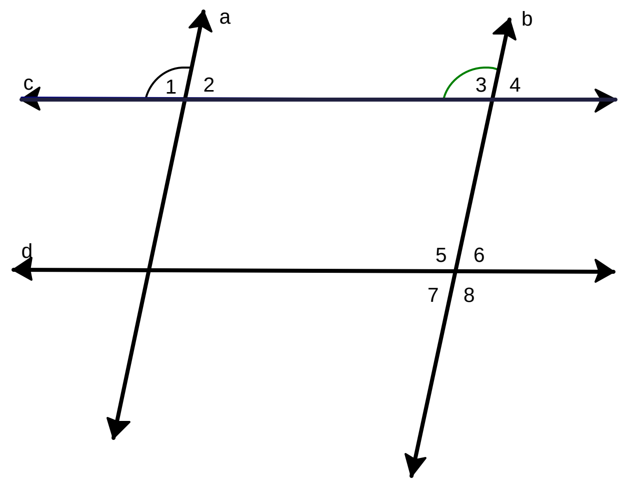
Proof:

<u>Exercise #7</u>
(2.3: Example 1)

Which lines are parallel if

a) $\angle 1 \cong \angle 3$?

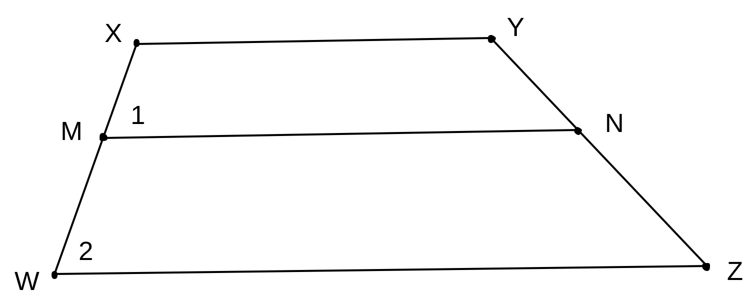
b) $\angle 3 \cong \angle 8$



<u>Exercise #8</u>
(2.3: #22)

Given: $\overline{XY} \parallel \overline{WZ}$
$\angle 1 \cong \angle 2$

Prove: $\overline{MN} \parallel \overline{XY}$



Proof: