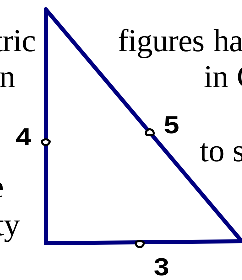
Elementary Geometry
Introduction

Historical Note

Geometry and Numbers

The history of mathematics shows that numbers and geometric figures have long been linked together. Although **plane geometry** in its present form began in Greece during the years 650 – 300 B.C., the peoples of the earlier Babylonian and Egyptian civilizations (1800 – 650 B.C.) had used numbers and geometric figures to solve practical problems. Both groups knew, for example, that a **right angle** could be obtained by forming a **triangle** whose sides were 3, 4, and 5 units long. The ability to lay out a square corner in this way had many applications.

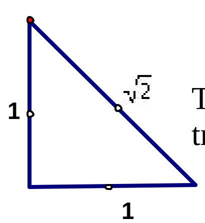


The early Egyptians and Babylonians did not content themselves with the 3-4-5 triangle. They knew that if the sides *a, b*, and *c* satisfied the equation $a^2 + b^2 = c^2$, the **angle** opposite side *c* (the hypotenuse) would be a right angle, but they were troubled by the simple case $1^2 + 1^2 = c^2$, or $2 = c^2$. Today we refer to the number *c* where $c^2 = 2$ as *the square root of 2* ($\sqrt{2}$).

However, for the Greeks of 650 – 300 B.C., not having "exact" numbers (i.e., integers or fractions) for the irrationals was a genuine and important problem. To overcome this difficulty they decided to work all numbers geometrically. They began with a certain length to represent 1. Other rational numbers were then represented in terms of this length.

Can you represent the number 2?

How about the number $\frac{1}{2}$? (☺ We are going to learn how to do this construction soon)



The irrational number $\sqrt{2}$ was represented by the **length** of the hypotenuse of a right triangle whose other two sides were each one unit long.

Arithmetic operations were done geometrically. The answer to $1+\sqrt{2}$, for example, was represented by the line segment formed by adjoining a segment representing 1 to a segment representing $\sqrt{2}$.

The answer to a multiplication of two numbers was represented by the area of a rectangle, and the product of three numbers was a volume. A product of four numbers, however, was inconceivable because there was no geometric figure to represent it.

The geometry of classical Greece was a masterpiece of mathematics, and it had a profound influence on the development of European mathematics for many hundreds years. Its effect, in a very small way, may be noted today in our practice of referring to the multiplication 5 x 5, for example, as "squaring" and to the multiplication of 5 x 5 x 5 as "cubing".

The Nature of Geometry

The word ***geometry*** comes from the Greek language and means, "**earth measure**", but the study of geometry involves more than just the size of our planet. Our world is filled with objects that have size, shape, and position and that are separated by varying distances, and geometry is a systematic study of these observable properties. Hence, "earth measure" should suggest such figures as triangles, rectangles, and circles and the use of numbers to measure their sizes.

Our textbook is primarily about plane geometry, that is, about figures that can be drawn on a flat surface. The first systematic study of the properties of plane figures was begun in Greece by **Thales** (640 – 546 B.C.), who had learned about geometry from the Egyptians.

Thales' most famous student was **Pythagoras,** whose name still identifies an important property of right triangles. There is a legend that Pythagoras wanted to see if he could teach someone geometry. After finding a somewhat reluctant student, Pythagoras agreed to pay him a penny for each theorem he learned. Because the student was very poor, he worked diligently. After a time, however, the student realized that he had become more interested in geometry than in the money he was accumulating. In fact, he became so intrigued with his studies that he begged Pythagoras to go faster, offering now to pay him back a penny for each theorem. Eventually Pythagoras got all his money back!

Perhaps the most creative mathematician of ancient Greece was **Archimedes** (287 – 212 B.C.), whose many achievements include a good estimate of the numerical relation, denoted p , between the circumference (length) of any circle and its diameter.

Probably the most famous of the Greek geometers was **Euclid** (330 – 275 B.C.), the first to systematically organize in book form the then-known facts of plane geometry. For this reason, the geometry we will study is often called *Euclidean geometry*. Euclid's method, now called a **deductive system**, has had a profound effect on the nature of scientific study, and his books, called *Elements*, are probably the most famous textbooks of all times.

What is there about geometry that is so fascinating? Geometry was the first system of ideas developed by man in which a few simple statements were assumed and then used to derive ones that are more complex. Such a system is called *deductive*. The beauty of geometry as a deductive system has inspired men in other fields to organize their ideas in the same way. Sir Isaac Newton's *Principia*, in which he tried to present physics as a deductive system, and the philosopher Spinoza's *Ethics* are especially noteworthy examples.

There are countless examples of ways in which geometric facts may be used. It is said that Thales was able to find the height of and Egyptian pyramid by measuring the length of its shadow. This involves the idea of representing the physical situation by a geometric figure or model. No doubt Thales knew enough about triangles to relate his measurements to a geometric model and determine the unknown height.

The study of geometry is also valuable because of its wide variety of applications to other subjects. Astronomers, for example, have used geometry to measure the distance from the earth to the moon, artists have used it to develop the theory of perspective, and chemists have used it to understand the structure of molecules.

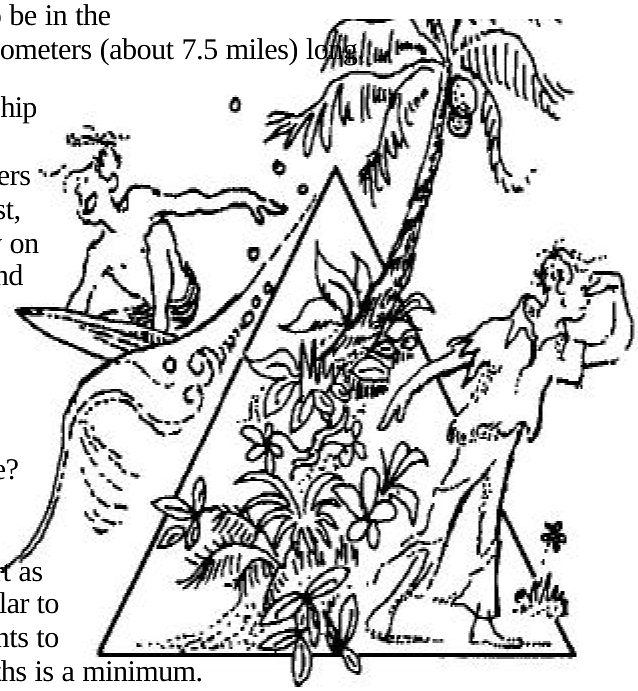
These examples point up the fact that to apply geometry we must learn about basic geometric figures containing **points, lines, angles, triangles, and so on.** It is these figures that are used to model physical reality and thus bridge the gap between the things we observe and the mathematical concepts we use to answer questions.
The relations and facts of geometry will be developed in the form of a mathematical system, a modern version of Euclid's approach.

The Puzzles of the Surfer and the Spotter

(Turn in some of your experiments and your *guess* with your first homework for extra credit).

One night a ship is wrecked in a storm at sea and only tow members of the crew survive. They manage to swim to a deserted tropical island where they fall asleep exhausted. After exploring the island the next morning, one of the men decides that he would like to stay there and spend the rest of his life surfing on the beaches. The other man, however, wants to escape and decides to use his time looking for a ship that might rescue him.

The island is overgrown with vegetation and happens to be in the shape of an **equilateral triangle,** each side being 12 kilometers (about 7.5 miles) long.



Wanting to be in the best possible position to spot any ship that might sail by, the man who hopes to escape (we will call him the “spotter”) goes to one of the corners of the island. Since he doesn’t know which corner is best, he decides to rotate from one to another, spending a day on each. He wants to build a shelter somewhere on the island and a path from it to each corner so that the sum of the lengths of the three paths is a minimum. (Digging up the vegetation to clear the paths is not an easy job). **Where should the spotter build his house?**

What about the surfer? Where should he build his house? He likes the beaches along all three sides of the island and decides to spend an equal amount of time on each. To make the paths from his house to each beach as short as possible, he constructs them so that they are perpendicular to the lines of the beaches. The surfer, like the spotter, wants to locate his house so that the sum of the lengths of the paths is a minimum. **Where is the best place on the island for him?**

To be confident of the conclusions that we may draw in geometry, we must have some basis for understanding them and for convincing others that they are correct. Deductive reasoning provides this basis.

Logic (Appendix B & Section 1.1)

One of the goals of studying geometry is to develop the ability to think critically. An understanding of the methods of deductive reasoning is fundamental in the development of critical thinking.

<u>Exercise</u>

The following passage is followed by a series of statements that are conclusions that might be drawn on the basis of accepting all of the information in the passage as literally true. Some of these conclusions are true, some are false, and some are questionable – that is, from the information in the passage, it cannot be definitely determined whether they are true or false. Mark each "true", "false", or "not certain".

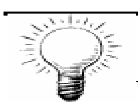
"In answer to your question what we got out of English so far I am answering that so far I got without a doubt nothing out of English. Teachers were sourcastic sourpuses or nervous wrecks. Half the time they were from other subjects or only subs….

Also no place to learn. Last term we had no desks to write only wet slabs from the fawcets because our English was in the Science Lab and before that we had no chairs because of being held in Gym where we had to squatt.

Even the regulars Mrs. Lewis made it so boreing I wore myself out yawning, and Mr. Loomis (a Math) hated teaching and us."

Bel Kaufman, *Up The Down Staircase*

1. The student writing this essay dislikes English. ____________________
2. The student writing this essay answered the question concerning what he (or she) had learned from his (or her) English class. ____________________
3. There are several spelling mistakes in this essay. ____________________
4. The faucets in the science room leaked. ____________________
5. The gym had no bleachers since the students had to squat on the floor.____________________
6. The school is overcrowded. ____________________
7. Mrs. Lewis was not one of the substitute teachers. ____________________
8. Mr. Loomis would rather teach math than English. ____________________



We need to have a common basis for drawing conclusions with which we can all agree. Our study of the nature of deductive reasoning will help provide this basis.

Definition A **STATEMENT** is a group of words and symbols that can be classified collectively as true or false.

Exercise #1 (1.1 - #1)

Which sentences are statements? If a sentence is a statement, classify it as true or false.

a) Where do you live? ______________________

b) $4+7\neq 5$ ______________________

c) Washington was the first U.S president. ______________________

d) $x+3=7$ when $x=5$. ______________________

Note: We represent statements by letters such as P, Q, and R.

Definition The **NEGATION** of a given statement P makes a claim opposite that of the original statement.

> If P is a statement, $\sim P$ (read "not P) indicates its negation.

Exercise #2 (1.1 - #3)

Give the negation of each statement.

a) Christopher Columbus crossed the Atlantic Ocean.

____

b) All jokes are funny. ______________________

c) $2+5=7$ ______________________

d) Some dogs can fly. ______________________

Definition A **TRUTH TABLE** is a table that provides the truth values of a statement by considering all possible true/false combinations of the statement's components.

P	$\sim P$
T	F
F	T

→ If P is true, then $\sim P$ is false.
→ When P is false, $\sim P$ is true.

COMPOUND STATEMENTS

Statements can be combined to form compound statements:

Definition A **CONJUNCTION** is a statement of the form ***P* and *Q*.**

$P \wedge Q$

P	Q	$P \wedge Q$

For the conjunction to be true, it is necessary for P to be true *and* Q to be true.

Exercise #3 (B1,Example 2)

Let P=”Babe Ruth played baseball” and Q=”4 + 3 <5.” Classify as true or false:

a) $P \wedge Q$ ____

b) $P \wedge \sim Q$ ____

Definition A **DISJUNCTION** is a statement of the form ***P* or *Q*.**

$P \vee Q$

P	Q	$P \vee Q$

A disjunction is false only if P and Q are both false.

Example: You can join the Math Club if you have an A average or you are enrolled in a mathematics class.

Exercise #4 (B1,Example 3)

Let P=”Babe Ruth played baseball” and Q=”4 + 3 <5.” Classify as true or false:

a) $P \vee Q$ ____

b) $P \vee \sim Q$ ____

Exercise #5 (B1 - #3, 4, 7)

Statement P is true, Q is true, and R is false. Classify each statement as true or false.

a) $P \wedge Q$ b) $Q \wedge R$ c) $P \wedge (Q \vee R)$

Definition An **IMPLICATION or CONDITIONAL** is a statement of the form **"If *P*, then *Q*."**

Note: *P* is called the *antecedent* (or *hypothesis)*
Q is called the *consequent* (or *conclusion)*

$$P \rightarrow Q$$

P	Q	$P \rightarrow Q$

The conditional statement makes a promise and fails to satisfy the conditions of this promise only when P is true and Q is false.

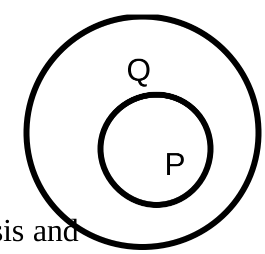
Example Consider the claim, "If you are good, then I'll give you a dollar."
The only way the claim is false is when "you are good, but I don't give you the dollar."

CONVERSE . INVERSE. CONTRAPOSITIVE

Lewis Carroll, the author of *Alice's Adventures in Wonderland* and *Through the looking Glass*, was a mathematician teacher who wrote stories as a hobby. His books contain many amusing examples of both good and deliberately poor logic. Consider the following conversation held at the Mad Hatter's tea Party.

" Then you should say what you mean,", the March Hare went on.
"I do,", Alice hastily replied; "at least – at least I mean what I say – that's the same thing, you know."
" Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
"You might just as well say, "added the March Hare, "that 'I like what I get' is the same as 'I get what I like'!"
"You might just as well say," added the Dormouse, who seemed to be talking in his sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"
"It *is* the same thing with you," said the Hatter, and here the conversation dropped, and the party went silent for a minute.

Carroll is playing here with pairs of related statements and the Hatter, the Hare, and the Dormouse are right: the sentences in each pair do not say the same thing at all.



Conditional statement: $P \rightarrow Q$ If P, then Q. (P implies Q)

Its **CONVERSE** $Q \rightarrow P$ If Q, then P.

- The converse of a conditional statement is formed by interchanging its hypothesis and conclusion.
- The converse of a true statement may be false. It is also possible that it may be true, but in either case a statement and its converse do not have the same meaning.

Its **INVERSE**: $\sim P \rightarrow \sim Q$ If *not* P, then *not* Q.

- The inverse of a conditional statement is formed by denying both its hypothesis and conclusion.

Its **CONTRAPOSITIVE**: $\sim Q \rightarrow \sim P$ If *not* Q, then *not* P.

- The contrapositive of a conditional statement is formed by interchanging its hypothesis and conclusion and denying both.

Exercise #6 (1.1 - #5, 6, 9)

Classify each statement as simple, conditional, a conjunction, or a disjunction.

a) If Alice plays, the volleyball team will win. ______________________

b) Alice played and the team won. ______________________

c) Matthew is playing shortstop. ______________________

Exercise #7 (1.1 - #11, 12, 16)

State the hypothesis and the conclusion of each statement.

a) If you go to the game, then you will have a great time.

Hypothesis: ____

Conclusion: ____

b) If two cords of a circle have equal lengths, then the arcs of the chords are congruent.

Hypothesis: ____

Conclusion: ____

c) Vertical angles are congruent when two lines intersect.

Hypothesis: ____

Conclusion: ____

Exercise #8

Identify the relationship of each of the lettered statements to the numbered statement if possible. Write “converse,” “inverse,” “contrapositive,” “original statement,” or “none,”, as appropriate.

1. Lady kangaroos do not need handbags.

a) If a kangaroo is not a lady, it needs a handbag. __________________

b) If it needs a handbag, then it is not a lady kangaroo. __________________

c) A kangaroo does not need a handbag if it is a lady. __________________

Exercise #9

Write the inverse, converse, and contrapositive of the following statement:

“If you live in Atlantis, then you need a snorkel.”

a) Inverse: ____

b) Converse: ____

c) Contrapositive: ____

<u>Definition</u> Two statements are **logically equivalent** if their truth values are the same for all possible true/false combinations of their components.

<u>Exercise #10</u>
(B1,Example 5)

Write the contrapositive of the given statement and then show, using a truth table, that the conditional statement is logically equivalent to its contrapositive.

" If I am sleeping, I am breathing."

Contrapositive: ____

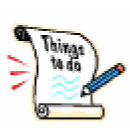
$$(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$$

<u>Definition</u> A TAUTOLOGIE is a statement that is true for all possible truth value of its components.

<u>Exercise #11</u>
(B1 - #15, 18)

Form a truth table and determine all possible truth values for the given statement. Is the given statement a tautology?

a) $(P \vee Q) \rightarrow P$



b) $[(P \rightarrow Q) \wedge P] \rightarrow Q$

DEMORGAN'S LAWS

In the study of logic, DeMorgan's Laws (19th century) are used to describe the negation of the conjunction ($\wedge$) and disjunction ($\vee$).

1. $\left[\sim(P \wedge Q)\right] \leftrightarrow \left[\sim P \vee \sim Q\right]$ The negation of a conjunction is the disjunction of negations.

2. $\left[\sim(P \vee Q)\right] \leftrightarrow \left[\sim P \wedge \sim Q\right]$ The negation of a disjunction is the conjunction of negations.

Proof of DeMorgan's first law

Proof of DeMorgan's second law (B1 - #25)

Exercise #12
(B1, #19, 21, 23)

Use DeMorgan's Laws to write the negation of the given statement.

a) $P \wedge Q$

b) Mary is an accountant or hamburgers are health food.

c) It is cold and snowing.

Exercise #13
(B1 - #29)

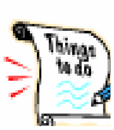
Use a truth table to show that $[P \wedge \sim Q]$ is the negation of $P \rightarrow Q$.

$$\left[\sim(P \rightarrow Q)\right] \leftrightarrow [P \wedge \sim Q]$$

Exercise #14
(B1 - #30, 31)

Write the negation of the given statement.

a) If it is medicine, then it tastes bad.



b) If I am good, then I can go to the movie.

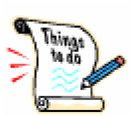
VALID ARGUMENTS

Suppose that during a trial a lawyer claims that, from the evidence presented, the guilty person is obviously color-blind and that everyone on the jury accepts this as true. Then he produces proof that Mr. Black is color-blind. Must the jury conclude that Mr. Black is guilty? Suppose also that it is established that Miss White is not color-blind. Must Miss White be innocent?

Definition An **ARGUMENT** is a set of statements called **premises**, followed by a statement called the **conclusion.** In a **VALID ARGUMENT**, the truth of the premises forces a conclusion that must also be true.

LAW OF DETACHMENT

$1. P \to Q$	Premise 1
$2. P$	Premise 2
C. $\boldsymbol{Q}$	Conclusion



Give the symbolic form and prove the Law of Detachment

ATTENTION!!! INVALID ARGUMENT

$1. P \to Q$	Premise 1
$2. Q$	Premise 2
C. P	Conclusion

Exercise #15
(B2 - #1,2)

Use the Law of Detachment to draw a conclusion.

a) If two angles are complementary, the sum of their measures is 90^{o}. $\angle 1$ and $\angle 2$ are complementary.

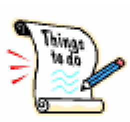
CONCLUSION: ____

b) If it gets hot this morning, we will have to turn on the air conditioner. It is hot this morning.

CONCLUSION: ____

LAW OF NEGATIVE INFERENCE

$1. P \rightarrow Q$	Premise 1
$2. \sim Q$	Premise 2
C. $\sim P$	Conclusion



Give the symbolic form and prove the Law of Negative Inference

Exercise #16
(B2 - #7,8)

Use the Law of Negative Inference to draw a conclusion.

a) If Tom doesn't finish the jobm then I will not pay him. I did pay Tom for the job.

CONCLUSION: ____

c) If the traffic light changes, then you can travel through the intersection. You cannot travel through the intersection.

CONCLUSION: ____

LAW OF SYLLOGISM

1. $P \to Q$	Premise 1
2. $Q \to R$	Premise 2
C. $P \to R$	Conclusion

Give the symbolic form and prove the Law of Syllogism

Exercise #17
(B2 - #9)

Use the Law of Syllogism to draw a conclusion.

a) If Izzi lives in Chicago, then she lives in Illionois. If a person lives in Illinois, then she lives in the Midwest.

CONCLUSION: ______________________________

Exercise #18
(B2 - #15, 16)

Determine which arguments are valid.

a) 1. If Bill and Mary stop to visit, I'll prepare a meal.
2. Bill stopped to visit at 5 p.m.

C. I prepared a meal.

VALID: YES NO

b) 1. If it turns cold and snows, I'll build a fire in the fireplace.
2. The temperature began to fall around 3 p.m.
3. It began snowing before 5 p.m.

C. I built a fire in the fireplace.

VALID: YES NO

LAW OF DENIAL (B2 - #21)

1. $P \vee Q$	Premise 1
2. $\sim Q$	Premise 2
C. P	Conclusion

Proof of the Law of Denial

Exercise #19 (B2 - # 22)

Use the Law of Denial to draw a conclusion.

Terry is sick or hurt.
Terry is not hurt.

CONCLUSION: ____

REASONING

1. INTUITION – With intuition, a sudden insight allows one to make a statement without applying any formal reasoning.

2. INDUCTION – Using specific observations and experiments to draw a general conclusion.
3. DEDUCTION – The type of reasoning in which the knowledge and acceptance of selected assumptions guarantees the truth of a particular conclusion.

Exercise #20 (1.1 - #25, 26)

Name the type of reasoning (if any) used.

a) While participating in an Easter egg hunt, Sarah notices that each of the seven eggs she has found is numbered. Sarah concludes that all eggs used for the hunt are numbered.

Intuition ___ Induction ___ Deduction ___

b) You walk into your geometry class, look at the teacher, and conclude that you will have a quiz today.

Intuition ___ Induction ___ Deduction ___

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Harold R. Jacobs, Geometry, W.H. Freeman and Company, 1974