TEST 3 @ 140 points

Show your work for credit. Write all responses on separate paper. Please write only on one side and clearly label the exercises.

1. a) Graph $f(x)=2^{x-3}-1$. Label the axes and the points used. You can use transformations of functions (vertical and horizontal shifting, reflection about the axes, vertical or horizontal stretching or shrinking) or plotting points.

   b) What is the domain of the function? c) What is the range?

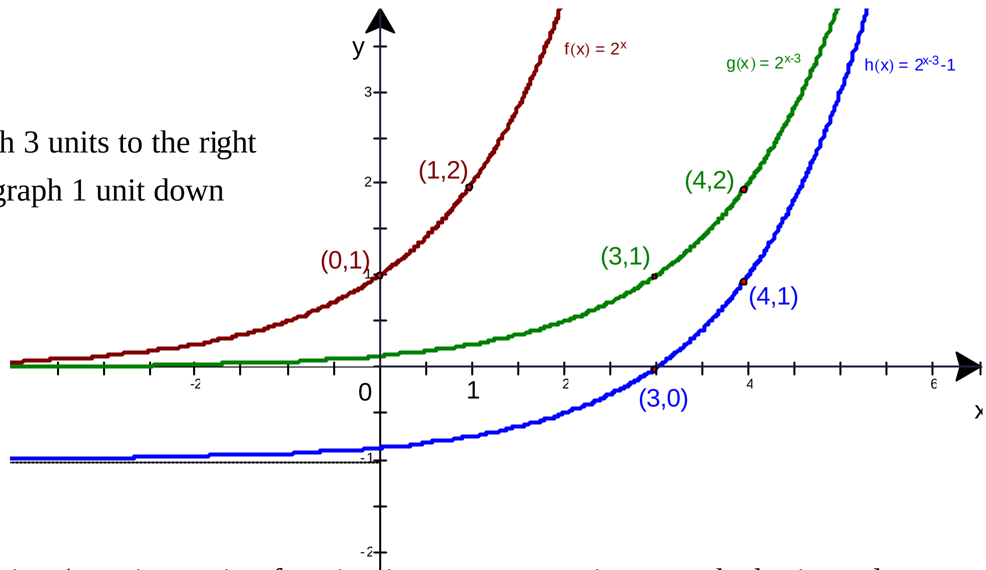
   d) What are the exact x-and y- intercepts?

   e) What kind of asymptote does the graph of *f* have? What is its equation?

   f) Is *f* increasing or decreasing? g) Does *f* have an inverse? Explain.

   h) Find $f^{-1}(x)$. i) What is the domain of f^{-1}? j) What is the range of f^{-1}?

   k) Explain how you can obtain the graph of f^{-1} from the graph of f. Graph f^{-1} on the same coordinate system as *f*.



a) 1st : $y=2^x$

2nd: $y=2^{x-3}$ shift the previous graph 3 units to the right

3rd: $y=2^{x-3}-1$ shift the previous graph 1 unit down

b) Domain f : $x\in\mathbb{R}$

c) Range f : $y\in(-1,\infty)$

d) $x-\cap:(3,0)$;

$y-\cap: x=0, y=2^{-3}-1$, $\left(0,-\frac{7}{8}\right)$

e) Horizontal asymptote: $y=-1$

f) increasing;

g) Yes, because it is a one-to-one function (any increasing function is one- to –one; it passes the horizontal line test);

h)

$y=2^{x-3}-1$

$y+1=2^{x-3}$

$\log_2(y+1)=(x-3)$

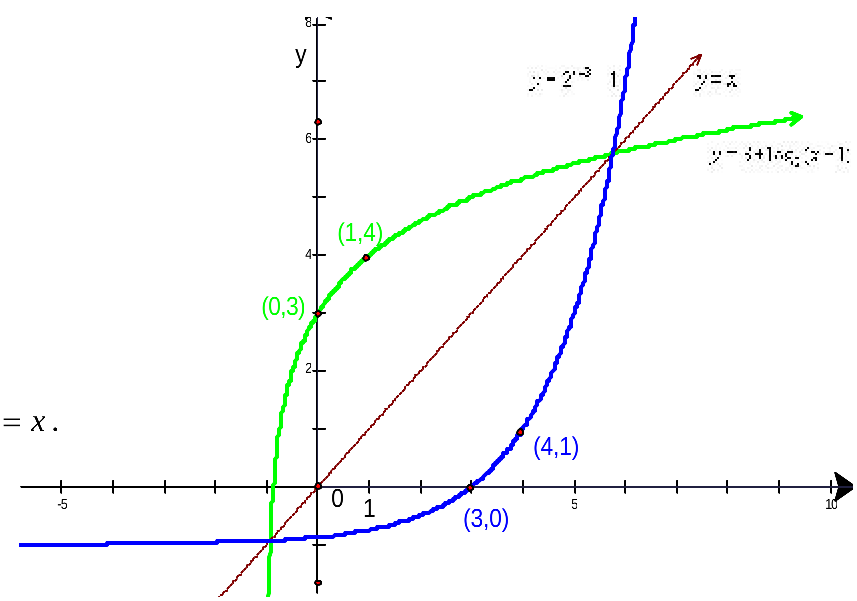
$x=3+\log_2(y+1)$

$f^{-1}(x)=3+\log_2(x+1)$

i) Domain f^{-1} : $x\in(-1,\infty)$

j) Range f^{-1} : $y\in\mathbb{R}$

k) The graph of f^{-1} is obtained by reflecting the graph of f about the line $y=x$.



2. a) Graph $f\left(x\right)=-\log_3(x-1)+2$. You can use transformations of functions (vertical and horizontal shifting, reflection about the axes, vertical or horizontal stretching or shrinking) or plo tting points.

b) What is the domain of the function? c) What is the range?

d) What are the exact x-and y- intercepts?

e) What kind of asymptote does the graph of f have? What is its equation?

f) Is f increasing or decreasing? g) Does f have an inverse? Explain.

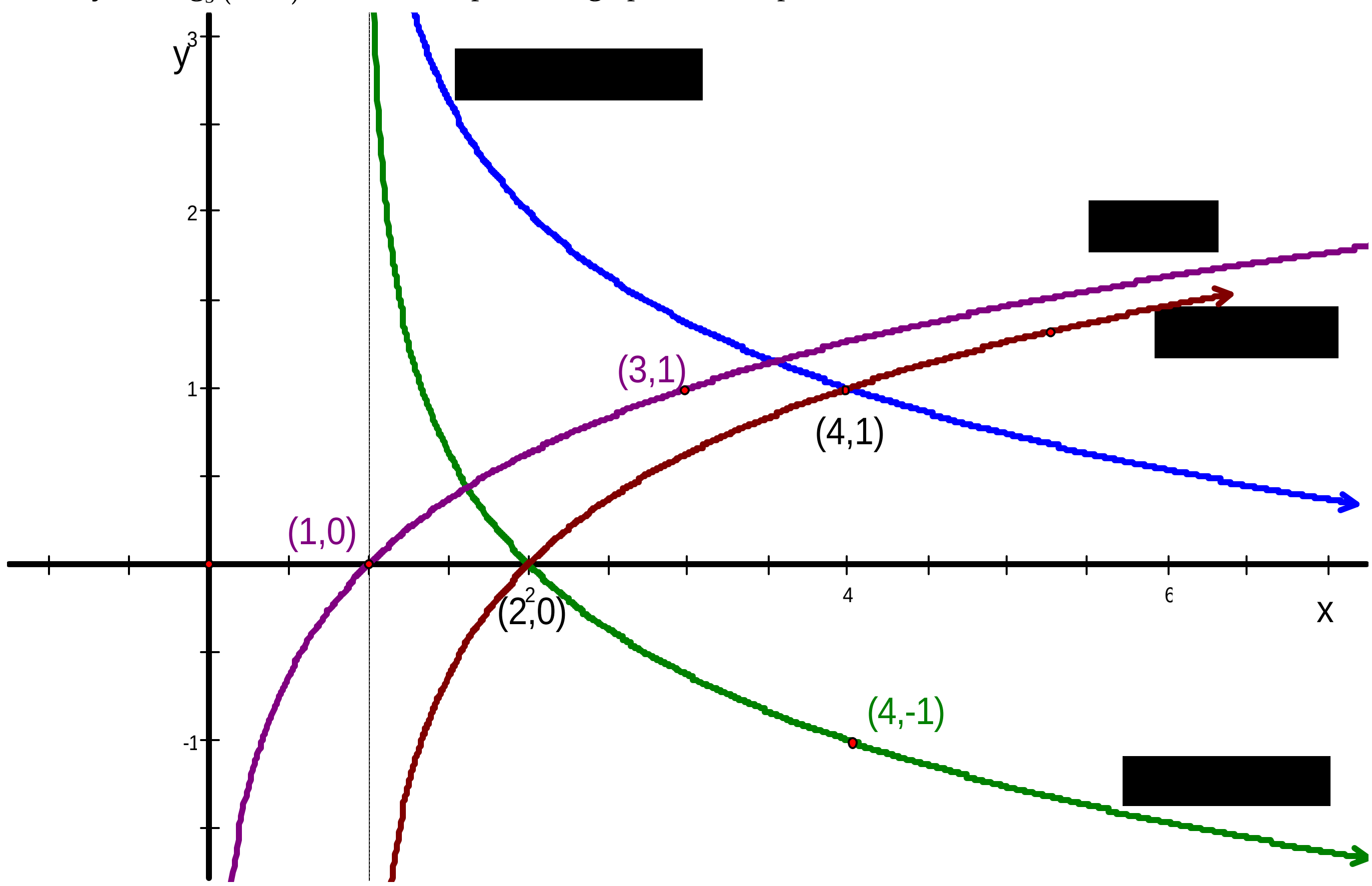
h) Find $f^{-1}\left(x\right)$. i) What is the domain of f^{-1}? j) What is the range of f^{-1}

a) 1st: $y=\log_3 x$

2nd: $y=\log_3\left(x-1\right)$ shift the previous graph 1 unit to the right

3rd: $y=-\log_3\left(x-1\right)$ reflect the previous graph about the x-axis

4th: $y=-\log_3\left(x-1\right)+2$ shift the previous graph 2 units up



b) Domain $f: x\in(1,\infty)$;

c) Range $f: y\in\mathbb{R}$

d) $x-\bigcap$: $\log_3\left(x-1\right)=2$

$x-1=9$

$x=10$

(10,0)

$y-\bigcap$: none; x>1

e) Vertical asymptote: $x=1$

f) decreasing

g) yes, it is a one-to-one function (passes the horizontal line test)

h) $y=-\log_3(x-1)+2$

$\log_3(x-1)=2-y$

$x-1=3^{2-y}$; $x=1+3^{2-y}$; $f^{-1}\left(x\right)=1+3^{2-x}$

i) Domain f^{-1}: $x\in\mathbb{R}$

j) Range f^{-1}: $y\in(1,\infty)$

3. Solve the following equations. Give exact answers.

a) $3e^{3x} = 5e^{5x}$

$\ln\left(3e^{3x}\right) = \ln\left(5e^{5x}\right)$

$\ln 3 + \ln e^{3x} = \ln 5 + \ln e^{5x}$

$\ln 3 + 3x = \ln 5 + 5x$

$\ln 3 - \ln 5 = 2x$

$x = \dfrac{\ln 3 - \ln 5}{2}$ or $x = \ln\sqrt{\dfrac{3}{5}}$

b) $10^{2x} + 3\left(10^x\right) - 10 = 0$

Let $10^x = t$; then $t^2 + 3t - 10 = 0$

$\cancel{t_1 = -5}$ or $t_2 = 2$

$10^x = 2$

$\log 10^x = \log 2$

$x = \log 2$

c) $\log x + \log(x-3) = 1$

Condition: x>3

$\log\left(x(x-3)\right) = 1$

$x(x-3) = 10$

$x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0,\quad x = 5$ or $\cancel{x = -2}$

4. Simplify the following expressions. Assume all variables represent positive numbers.

a) $\log\left(x^2-9\right) - 2\left[\log(x+3) + 3\log x\right] = \log\left(x^2-9\right) - 2\left(\log(x+3) + \log x^3\right)$

$= \log\left(x^2-9\right) - 2\log\left(x^3(x+3)\right)$

$= \log\left(x^2-9\right) - \log\left(x^3(x+3)\right)^2$

$= \log\dfrac{(x-3)(x+3)}{x^6(x+3)^2} = \log\dfrac{x-3}{x^6(x+3)}$

b) $\log_7\left(7k + 5r^2\right)$ - can't be simplified

5. Suppose a certain radioactive substance has a half-life of 5 years. An object starts with 20 kg of the radioactive material.

a) How much of the radioactive material is left after 10 years?

b) The object can be moved safely when the quantity of the radioactive material is 0.1 kg or less. How much time must pass before the object can be moved?

Let t be the time, and W be the amount of radioactive material.

t	W
0	20
5	$20\left(\frac{1}{2}\right)$
10	$20\left(\frac{1}{2}\right)^2$
t	$20\left(\frac{1}{2}\right)^{\frac{t}{5}}$

$W(t) = 20\left(\dfrac{1}{2}\right)^{\frac{t}{5}}$

a) $W(10) = 20\left(\dfrac{1}{2}\right)^{\frac{10}{5}} = 5kg$

b) $0.1 = 20\left(\dfrac{1}{2}\right)^{\frac{t}{5}}$

$0.005 = \left(\dfrac{1}{2}\right)^{\frac{t}{5}}$

$\ln 0.005 = \ln\left(\dfrac{1}{2}\right)^{\frac{t}{5}}$

$5\ln 0.005 = t\ln 0.5$

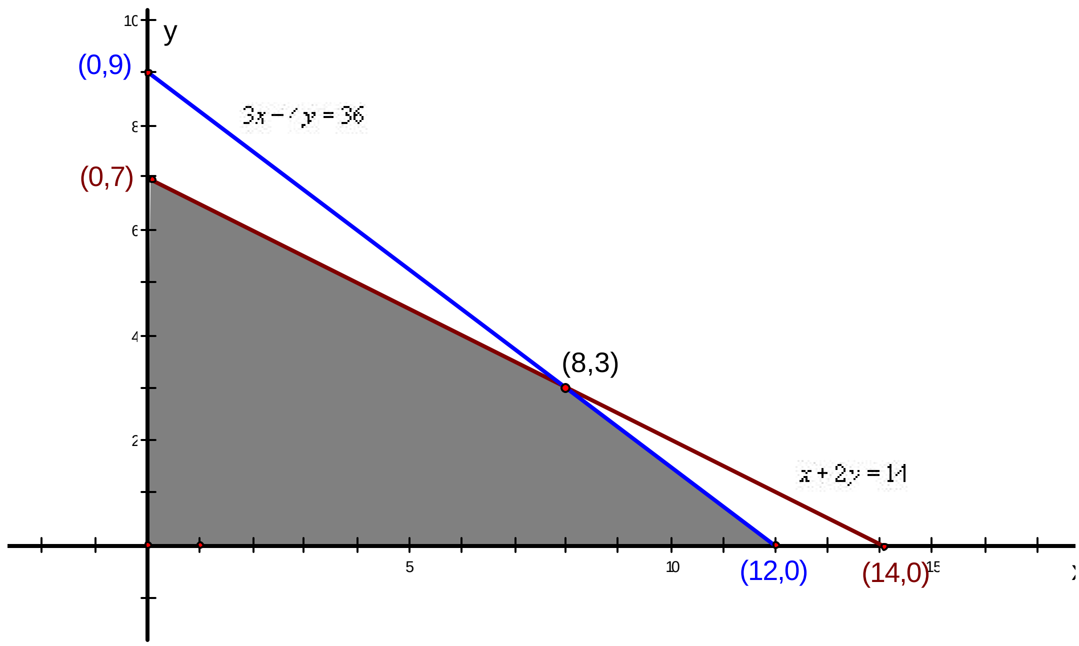
$t = \dfrac{5\ln 0.005}{\ln 0.5}$

$t = 38.2\,years$

6. An office manager wants to buy some filing cabinets. He knows that cabinet A costs $10 each, requires 6 ft^2 of floor space, and holds 8 ft^3 of files. Cabinet B costs $20 each, requires 8 ft^2 of floor space, and holds 12 ft^3. He can spend no more than $140 due to budget limitations, and his office has room for no more than 72 ft^2 of cabinets. He wants to maximize storage capacity within the limits imposed by funds and space. How many of each type of cabinet should he buy?

Let x be the number of type A cabinets, and y be the number of type B cabinets.

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 10x + 20y \leq 140 \\ 6x + 8y \leq 72 \end{cases}$$



$10x + 20y \leq 140$

$x + 2y \leq 14$

Boundary line: x+2y=14 has intercepts (0,7) and (14,0).

$6x + 8y \leq 72$

$3x + 4y \leq 36$

Boundary line 3x+4y=36 has intercepts (0,9) and (12,0).

The intersection between x+2y=14 and 3x+4y=36 is (8,3).

Vertices	V=8x+12y (the storage capacity)
(0,0)	0 ft^3
(12,0)	96 ft^3
(8,3)	100 ft^3
(0,7)	84 ft^3

Therefore, in order to maximize the Storage capacity, he needs to buy 8 type A cabinets and 3 type B cabinets.

7. Solve the following system of equations:

$$\begin{cases} x^2 - xy + y^2 = 5 \\ 2x^2 + xy - y^2 = 10 \end{cases}$$

If we add the equations terms by term,
$3x^2 = 15$, $x^2 = 5$, $x = \pm\sqrt{5}$

If $x = \sqrt{5}$, then $5 - \sqrt{5}y + y^2 = 5$, $y\left(y - \sqrt{5}\right) = 0$, $y = 0$ or $y = \sqrt{5}$

If $x = -\sqrt{5}$, then $5 + \sqrt{5}y + y^2 = 5$, $y\left(\sqrt{5} + y\right) = 0$, $y = 0$ or $y = -\sqrt{5}$.

Therefore, the solution set is $\left\{\left(\sqrt{5}, 0\right), \left(\sqrt{5}, \sqrt{5}\right), \left(-\sqrt{5}, 0\right), \left(-\sqrt{5}, -\sqrt{5}\right)\right\}$.

8. Find the partial fraction decomposition for the given rational expression: $\dfrac{x}{x^2 + 4x - 5}$

$$\frac{x}{x^2 + 4x - 5} = \frac{x}{(x+5)(x-1)} = \frac{A}{x+5} + \frac{B}{x-1}$$

$$x = A(x-1) + B(x+5)$$
$$x = (A+B)x - A + 5B$$
$$\begin{cases} A + B = 1 \\ -A + 5B = 0 \end{cases}$$

$$A = \frac{5}{6}, \; B = \frac{1}{6}$$

$$\frac{x}{x^2 + 4x - 5} = \frac{5}{6(x+5)} + \frac{1}{6(x-1)}$$