

3.5 Graphs of Rational Functions

Example 1

Graph the reciprocal function

$$f(x) = \frac{1}{x}$$

Answer the following questions:

a) What is the domain of the function?

$$x \in \mathbb{R} \setminus \{0\}$$

b) What is the range of the function?

$$y \in \mathbb{R} \setminus \{0\}$$

c) What are the x- and y-intercepts?

none

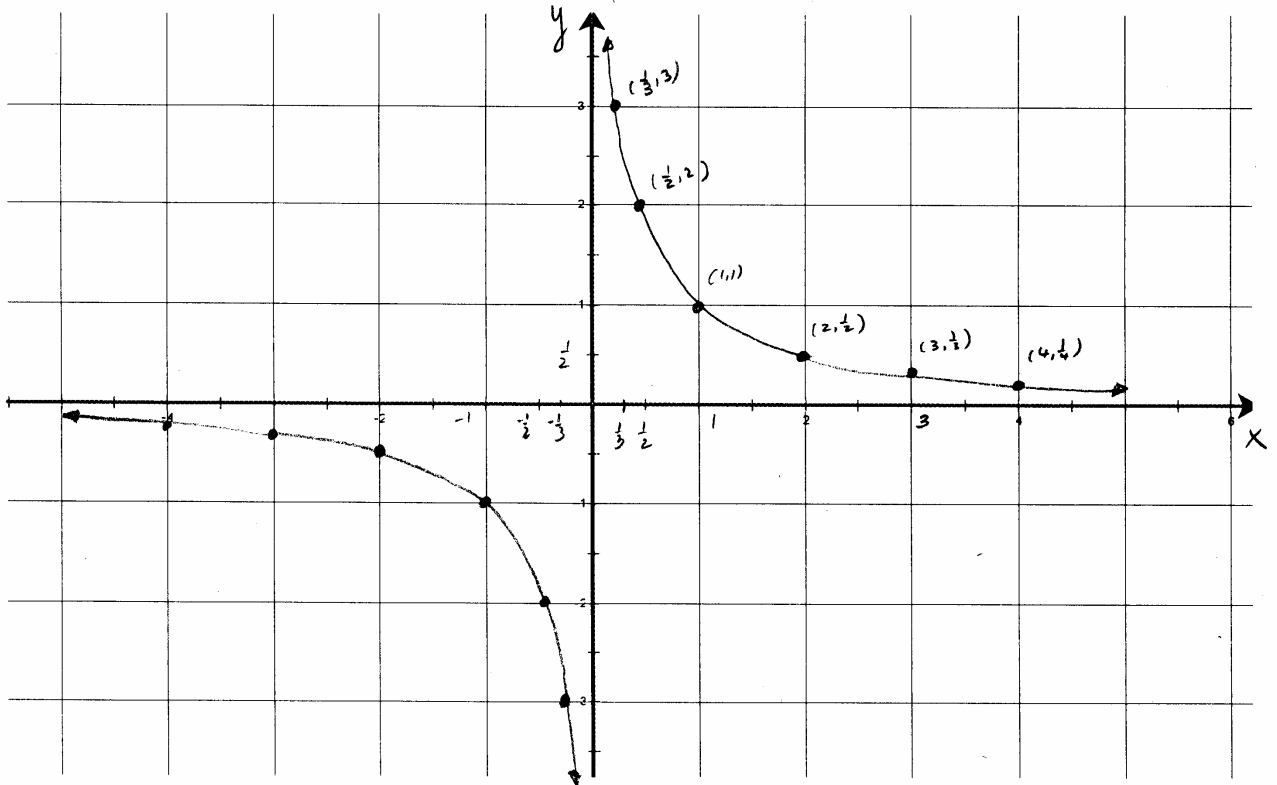
d) What is the end-behavior of the function, that is, what happens with the values of y as x goes to ∞ and $-\infty$?

$$\text{when } x \rightarrow \infty, f(x) \rightarrow 0; \text{ when } x \rightarrow -\infty, f(x) \rightarrow 0$$

e) What is the behavior of the function when x approaches 0?

$$\text{when } x \rightarrow 0^+, f(x) \rightarrow \infty; \text{ when } x \rightarrow 0^-, f(x) \rightarrow -\infty$$

x	$-\infty$	-4	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4	∞	
$f(x)$	0	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	$-\infty$	∞	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	0
	$y=0$ H.A.								$x=0$ V.A.					$y=0$ H.A.



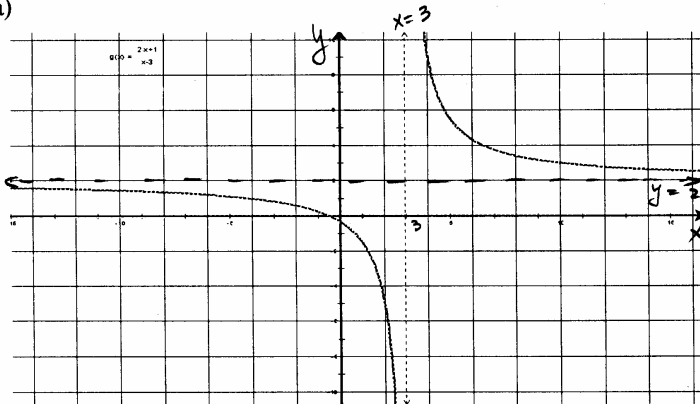
Definition A rational function is a function f of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$.

- Notations:
- $x \rightarrow \infty$ x approaches infinity (x increases without bound)
 - $x \rightarrow -\infty$ x approaches negative infinity (x decreases without bound)
 - $x \rightarrow a^+$ x approaches a from the right
 - $x \rightarrow a^-$ x approaches a from the left

Definition The line $x = a$ is a vertical asymptote for the graph of $f(x)$ if, when $x \rightarrow a$, $y \rightarrow \pm\infty$.
 The line $y = b$ is a horizontal asymptote for the graph of $f(x)$ if, when $x \rightarrow \pm\infty$, $y \rightarrow b$.

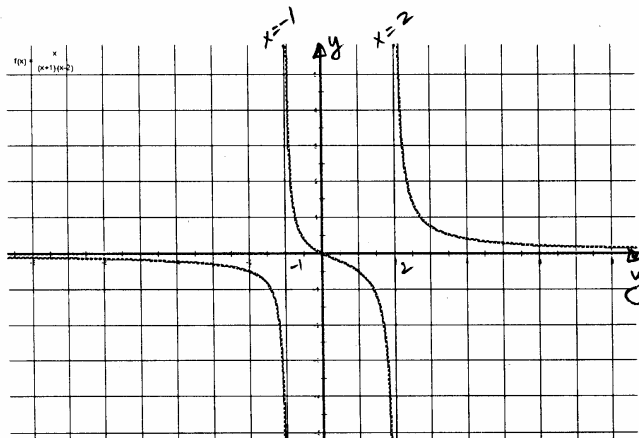
Exercise #1 Identify all the vertical and horizontal asymptotes of the following graphs. How can the vertical asymptotes be found? What about the horizontal asymptotes?

a)

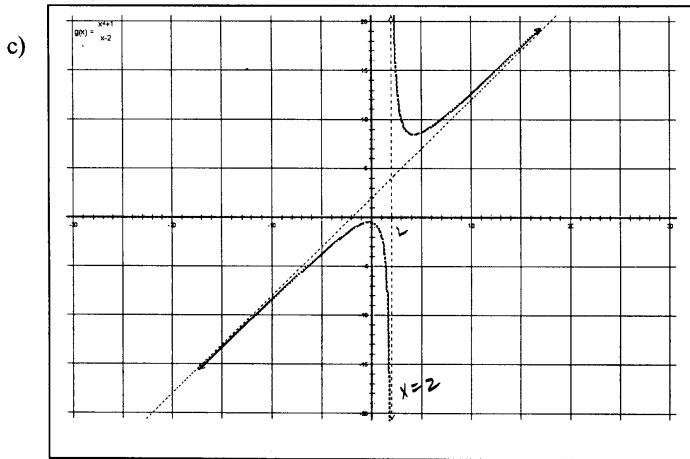


$f(x) = \frac{2x+1}{x-3}$
 V.A. $x=3$ because when $x \rightarrow 3$, $y \rightarrow \pm\infty$
 $x=3$ is a zero of the denominator.
 $f(x)$ is not defined in $x=3$
 H.A. $y=2$ because when $x \rightarrow \pm\infty$, $y \rightarrow 2$
 degree numerator = degree denominator

b)



$f(x) = \frac{x}{(x+1)(x-2)}$
 V.A. $x=2$ because when $x \rightarrow 2$, $y \rightarrow \pm\infty$
 $x=-1$ because when $x \rightarrow -1$, $y \rightarrow \pm\infty$
 $x=2$ and $x=-1$ are zeros of the denominator
 H.A. $y=0$ because when $x \rightarrow \pm\infty$, $y \rightarrow 0$
 degree numerator < degree denominator



$f(x) = \frac{x^2+1}{x-2}$
 V.A. $x=2$ because when $x \rightarrow 2, y \rightarrow \pm\infty$
 $x=2$ is a zero of the denominator
 H.A. - none
 degree numerator > degree denominator
 Oblique asymptote
 when $x \rightarrow \pm\infty, y \rightarrow$ oblique asymptote

Asymptotes for a rational function $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}$

1. The vertical asymptotes are the lines $x = c$, where c is a zero of the denominator.

2. If $n < m$, then $y = 0$ (the x-axis) is the horizontal asymptote.

If $n = m$, then $y = \frac{a_n}{b_n}$ is the horizontal asymptote.

If $n > m$, there are no horizontal asymptotes.

If, however, $n = m + 1$, then there is an oblique asymptote. Divide the numerator by the denominator and disregard the remainder.

$y = \text{quotient}$ is the oblique asymptote

Exercise #2 Identify all the asymptotes for the following functions:

$f(x) = \frac{2x+7}{x-5}$
 V.A. $x=5$
 H.A. $y = \frac{2}{1} = 2$
 no oblique asymptote

$g(x) = \frac{4x^2+x-5}{2x^2-3x-5}$
 $g(x) = \frac{(4x+5)(x-1)}{(2x-5)(x+1)}$
 V.A. $x = \frac{5}{2}$ and $x = -1$
 H.A. $y = \frac{4}{2} = 2$
 no oblique asymptote

$h(x) = \frac{x^2+6}{x-3}$
 V.A. $x=3$
 oblique asymptote:

$$\begin{array}{r} x+3 \\ x-3 \overline{) x^2+6} \\ \underline{-x^2+3x} \\ 13x+6 \\ \underline{-13x+9} \\ 15 \end{array}$$

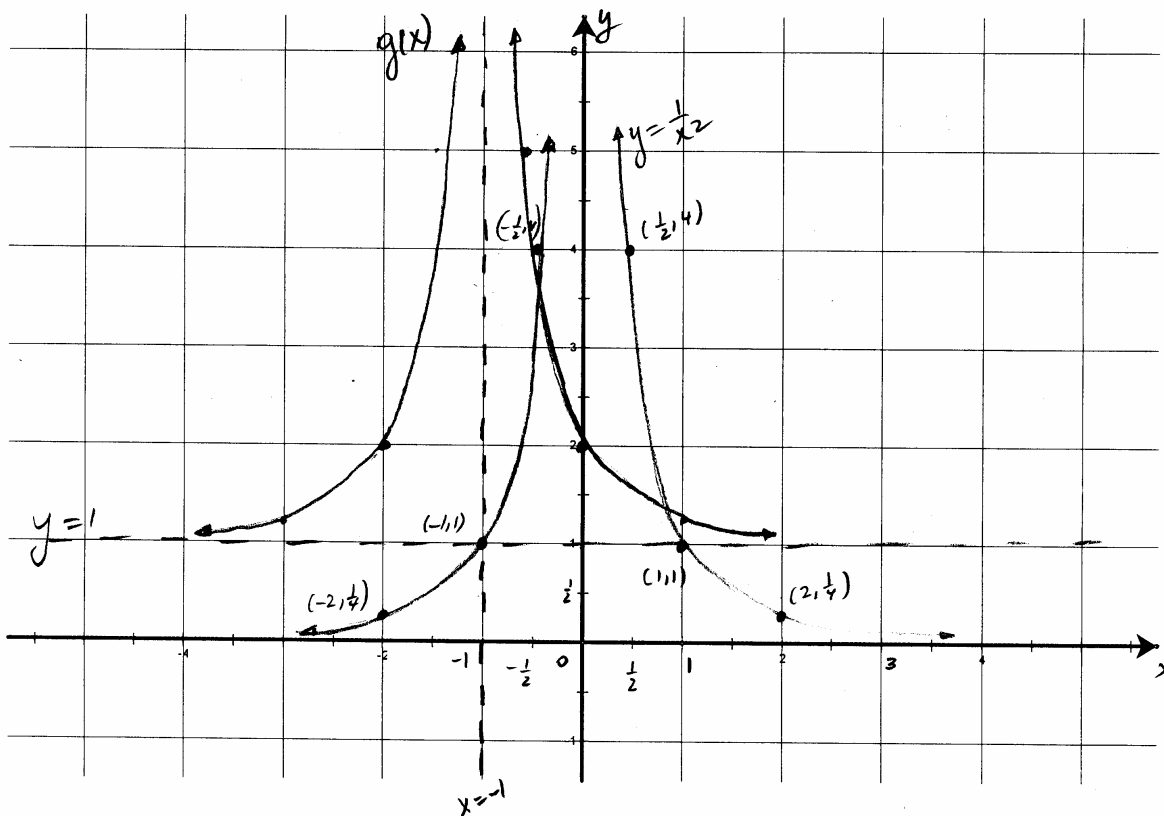
 $y = x+3$

$l(x) = \frac{1}{2x^2-2}$
 $l(x) = \frac{1}{2(x+1)(x-1)}$
 V.A. $x = -1$ and $x = 1$
 H.A. $y = 0$
 no oblique asymptote

Exercise #3 Graph the function $f(x) = \frac{1}{x^2}$. Find the domain, the asymptotes, and the x- and y-intercepts.

x	$-\infty$	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	∞
f(x)	0	$\frac{1}{9}$	$\frac{1}{4}$	1	$\frac{1}{4}$	∞	∞	1	$\frac{1}{4}$	$\frac{1}{9}$	0
	$y=0$ H.A.					$x=0$ V.A.					$y=0$ H.A.

no x-intercepts
 no y-intercepts
 graph is symmetric about the y-axis



Exercise #4 Show how to obtain the graph of $g(x) = \frac{1}{(x+1)^2} + 1$ from the graph of $f(x) = \frac{1}{x^2}$.

What are the asymptotes of $g(x)$?

- g(x)
- 1st start with $y = \frac{1}{x^2}$
 - 2nd $y = \frac{1}{(x+1)^2}$ shift left 1 unit
 - 3rd $y = \frac{1}{(x+1)^2} + 1$ shift up 1 unit

- V.A. $x = -1$
- H.A. $y = 1$

Exercise #5 Sketch the graph of $f(x) = \frac{x+1}{x-4}$. Find the domain, all the asymptotes, the x- and y-intercepts

Determine if the graph intersects its nonvertical asymptote. Plot additional test points, as needed.

x	$-\infty$	-2	-1	0	1	2	4	5	8	∞	
y	1	$-\frac{1}{6}$	0	$-\frac{1}{4}$	$-\frac{2}{3}$	$-\frac{3}{2}$	$-\infty$	∞	6	$\frac{9}{2}$	1

$y=1$ H.A. $x=4$ V.A. $y=1$ H.A.

x-n: $y=0 \Rightarrow x+1=0 \rightarrow x=-1$

y-n: $x=0 \Rightarrow y=-\frac{1}{4}$

H.A. $y = \frac{1}{1} = 1$ $f(x) = \frac{x(1+\frac{1}{x})}{x(1-\frac{4}{x})} = \frac{1+\frac{1}{x}}{1-\frac{4}{x}} \rightarrow 1$ when $x \rightarrow \pm\infty$

when $x \rightarrow 4^+$, $y \rightarrow \frac{5}{+0} = \infty$

$x \rightarrow 4^-$, $y \rightarrow \frac{5}{-0} = -\infty$

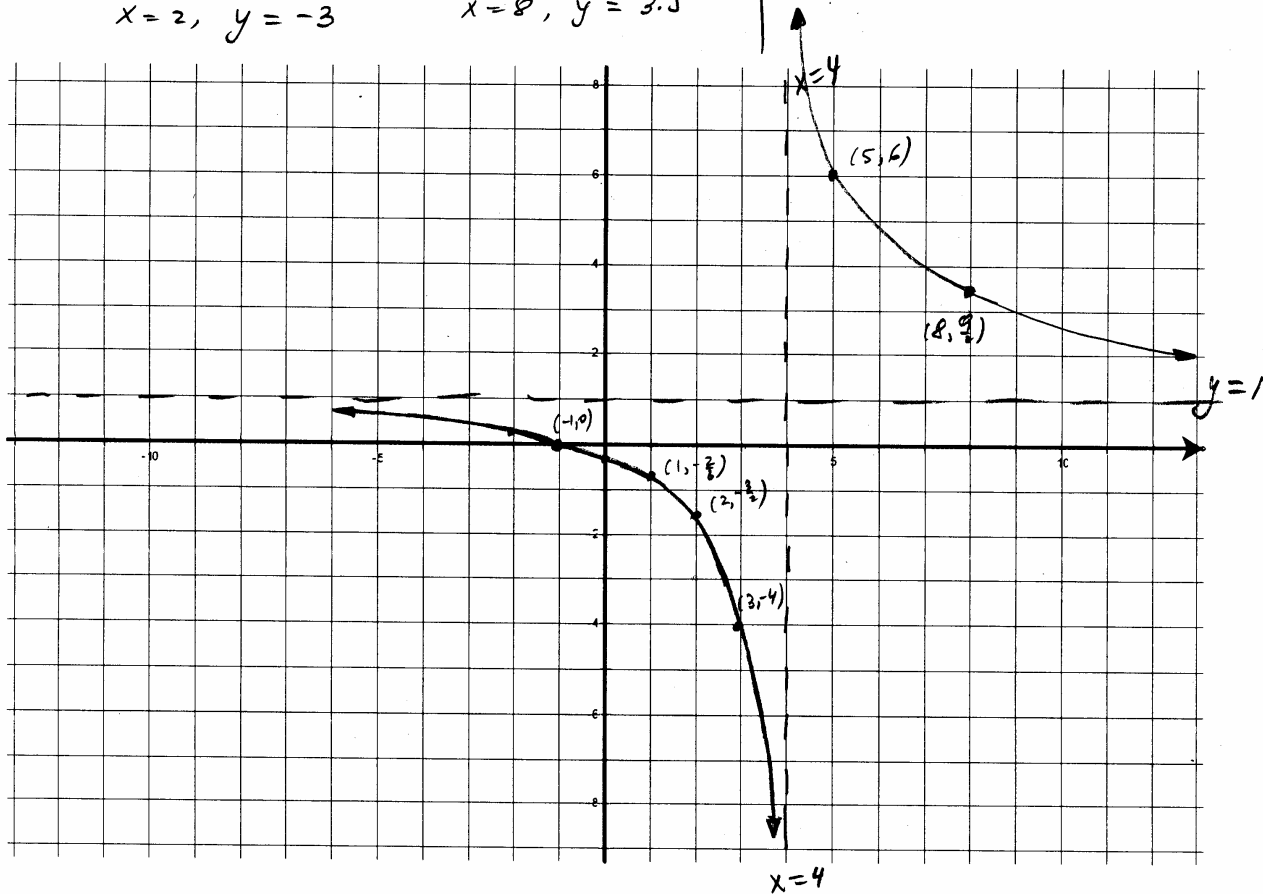
Test points $x=1, y = -\frac{2}{3}$
 $x=2, y = -3$

$x=5, y = 6$

$x=8, y = 3.5$

intersection of $f(x)$ with $y=1$

$\frac{x+1}{x-4} = 1 \Leftrightarrow x+1 = x-4$
 No solutions.



Exercise #6 Sketch the graph of $f(x) = \frac{x-2}{x^2-1}$. Find the domain, all the asymptotes, the x- and y-intercepts. Determine if the graph intersects its nonvertical asymptote. Plot additional test points, as needed.

x	$-\infty$	-2	-1	0	1	2	3	∞
f(x)	0	$-\frac{4}{3}$	$-\infty$ / ∞	2	∞ / $-\infty$	0	$\frac{1}{8}$	0
	$\boxed{y=0}$ H.A.		$\boxed{x=-1}$ V.A.		$\boxed{x=1}$ V.A.			$\boxed{y=0}$ H.A.

$f(x) = \frac{x-2}{(x+1)(x-1)}$ $x \neq 1, x \neq -1$

H.A. $y = 0$

x-n: $y = 0 \Leftrightarrow x - 2 = 0 \Leftrightarrow x = 2$

y-n: $x = 0 \Rightarrow y = \frac{-2}{-1} = 2$

$x \rightarrow -1^-$, $y \rightarrow \frac{-3}{(-0)(-2)} = -\infty$

$x \rightarrow -1^+$, $y \rightarrow \frac{-3}{(+0)(-2)} = \infty$

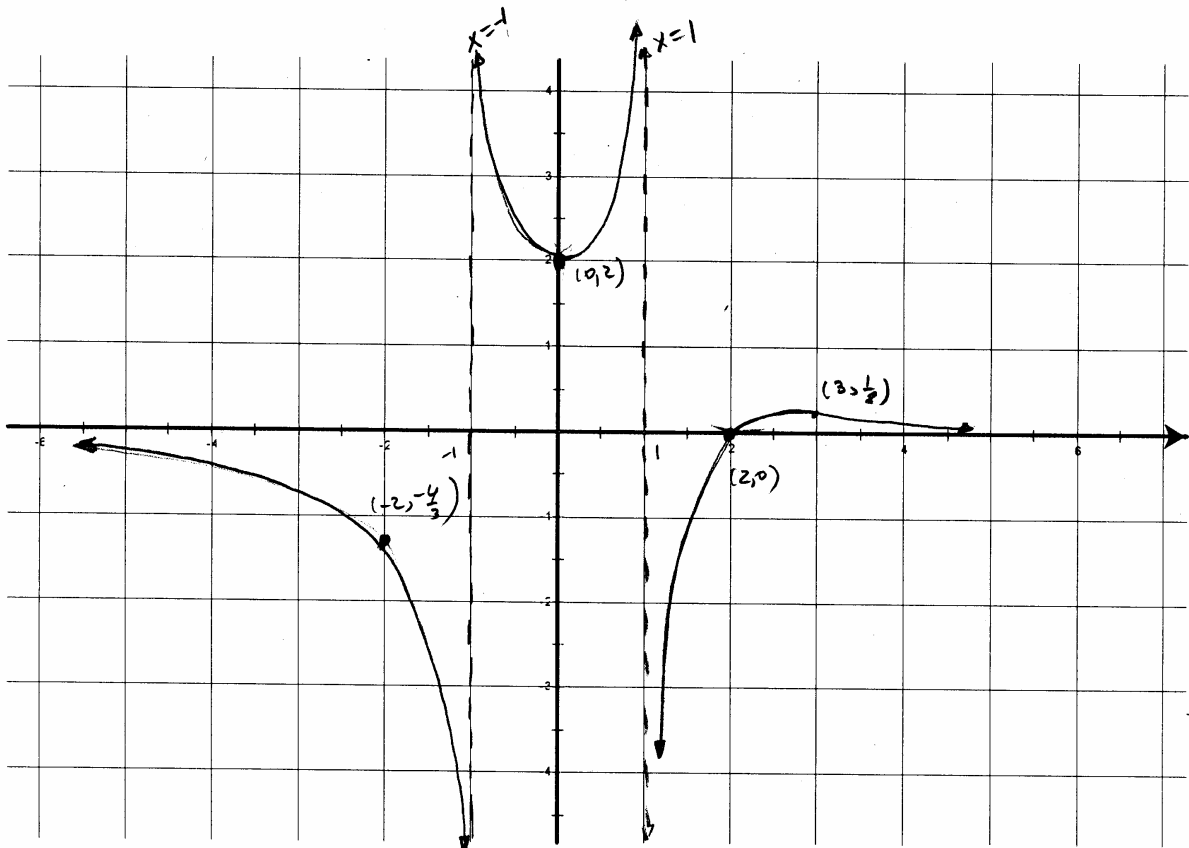
$x \rightarrow 1^-$, $y \rightarrow \frac{-1}{2(-0)} = \infty$
 $x \rightarrow 1^+$, $y \rightarrow \frac{-1}{2(+0)} = -\infty$

intersection with $y = 0$ $\frac{x-2}{x^2-1} = 0 \Leftrightarrow x = 2$

test points:

$x = 3, y = \frac{1}{8}$

$x = -2, y = \frac{-4}{3}$



Exercise #7 Sketch the graph of $f(x) = \frac{x^2 - 2x - 8}{x^2 - 4x + 3}$. Find the domain, all the asymptotes, the x- and y-intercepts. Determine if the graph intersects its nonvertical asymptote. Plot additional test points, as needed.

$$f(x) = \frac{(x-4)(x+2)}{(x-3)(x-1)} \quad x \neq 3, x \neq 1$$

x	$-\infty$	-2	-1	0	1	2	3	4	5.5	∞
f(x)	1	0	$-\frac{8}{3}$	$-\infty$	$+\infty$	8	$+\infty$	0	1	1
	$y=1$ H.A.				$x=1$ V.A.		$x=3$ V.A.			$y=1$ H.A.

H.A. $y = \frac{1}{1} = 1$

x-n: $y = 0 \Rightarrow x = 4, x = -2$

y-n: $x = 0, y = \frac{-8}{3} \approx -2.6$

$x \rightarrow 1^-$, $y \rightarrow \frac{(-3)(3)}{(-2)(-0)} = -\infty$

$x \rightarrow 1^+$, $y \rightarrow \frac{(-3)(3)}{(-2)(+0)} = +\infty$

$x \rightarrow 3^-$, $y \rightarrow \frac{(-1)(5)}{(-0)(2)} = +\infty$

$x \rightarrow 3^+$, $y \rightarrow \frac{(-1)(5)}{(+0)(2)} = -\infty$

Test points

$x = 6, y = \frac{2 \cdot 8}{3 \cdot 5} = \frac{16}{15} \approx 1.06$

$x = 2, y = \frac{(-2)(4)}{(-1)(1)} = 8$

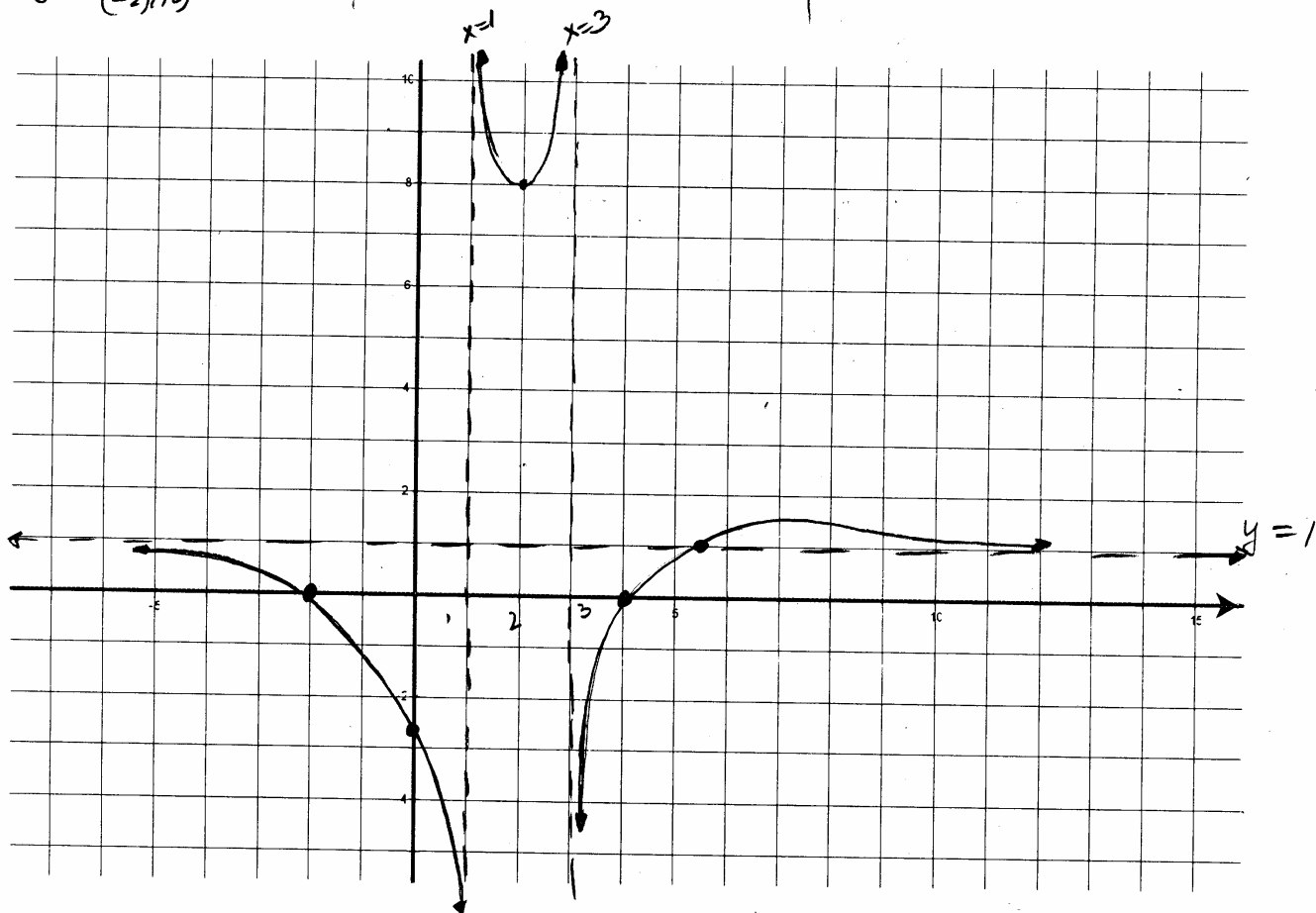
Intersection with $y = 1$

$$\frac{x^2 - 2x - 8}{x^2 - 4x + 3} = 1 \Rightarrow$$

$$x^2 - 2x - 8 = x^2 - 4x + 3$$

$$2x = 11$$

$$x = \frac{11}{2} \quad (5.5, 1)$$



Exercise #8 Sketch the graph of $f(x) = \frac{x^2+1}{x+3}$. Find the domain, all the asymptotes, the x- and y-intercepts

Determine if the graph intersects its nonvertical asymptote. Plot additional test points, as needed.

x	-∞	-4	-3	-2	-1	0	1	2	∞
	$y = x-3$ O.A.	-9	-∞	+∞	5	1	$\frac{1}{3}$	$\frac{1}{2}$	1
			$x = -3$ V.A.						$y = x-3$ O.A.

$x \neq -3$

no horizontal asymptote

Oblique asymptote:

$$\begin{array}{r} x-3 \\ x+3 \overline{) x^2+1} \\ \underline{-x^2-3x} \\ -3x+1 \end{array}$$

$y = x-3$

x	y
0	-3
3	0

x-n: $y=0$ no x-n

y-n: $x=0, y = \frac{1}{3}$

$x \rightarrow -3^-, y \rightarrow \frac{+10}{-0} = -\infty$

$x \rightarrow -3^+, y \rightarrow \frac{+10}{+0} = +\infty$

Test points

$x=2, y=1$

$x=-4, y=-9$

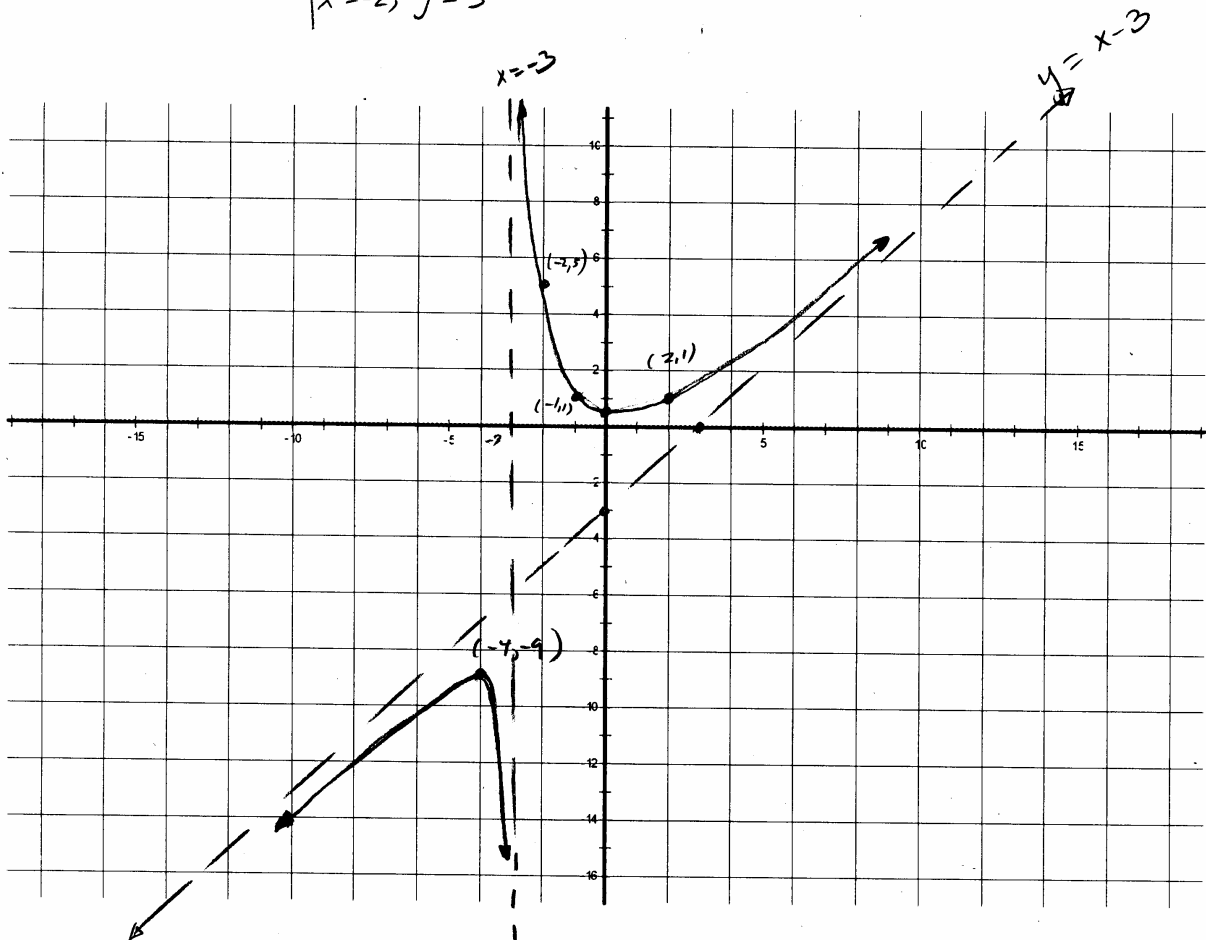
$x=-1, y = \frac{2}{2} = 1$

$x=-2, y=5$

interaction with $y = x-3$

$$\frac{x^2+1}{x+3} = x-3 \iff x^2+1 = x^2-9$$

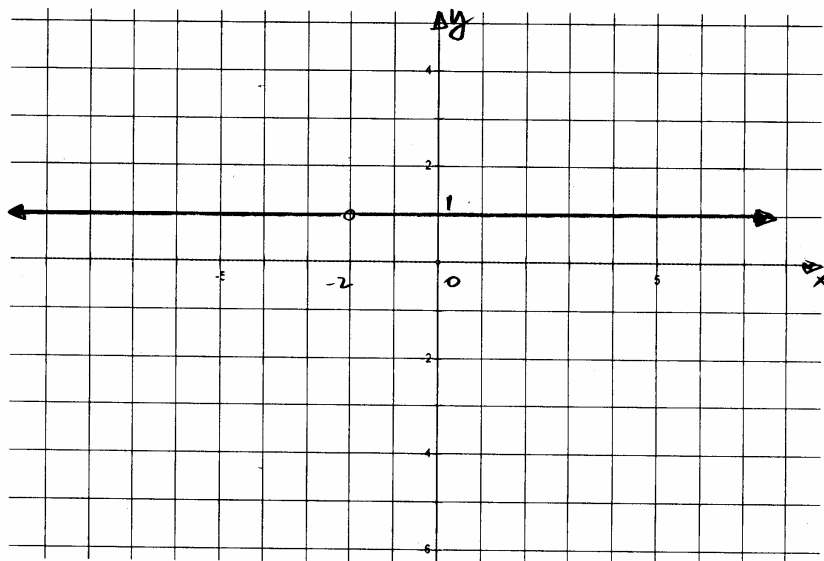
no solution



Exercise #9 Graph the following functions: $f(x) = \frac{x+2}{x+2}$ and $g(x) = \frac{x^2-9}{x+3}$.

$$f(x) = \frac{x+2}{x+2} = 1$$

$$x \neq -2$$



$$g(x) = \frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3}$$

$$x \neq -3$$

$$g(x) = x - 3$$

x	y
0	-3
3	0

