Section 3.1
Quadratic Functions and Models

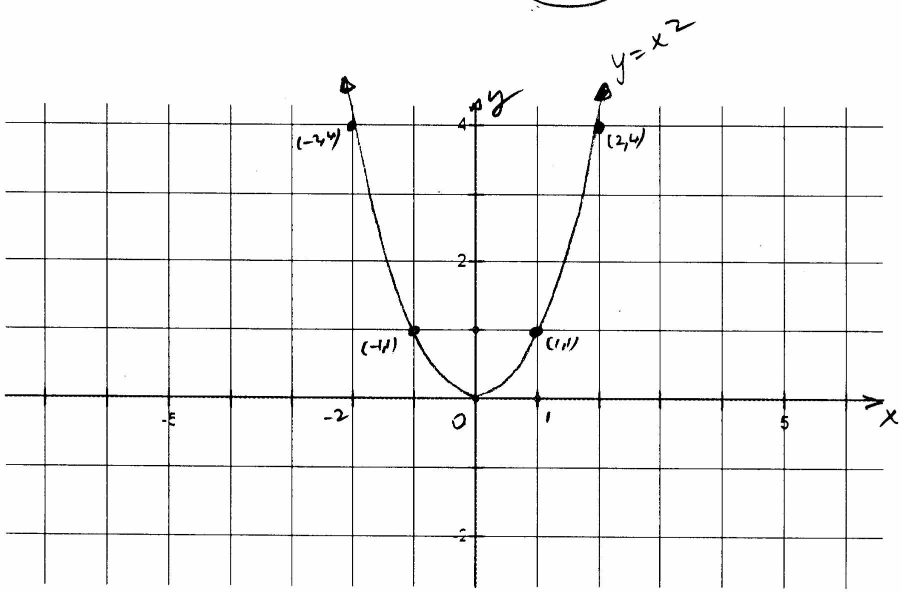
Quadratic Function: $f(x) = ax^2 + bx + c \quad (a \neq 0)$

The graph of a quadratic function is called a **parabola.**

Graphing Parabolas: Special Cases

The "basic" parabola is the graph of the simplest quadratic function $y = x^2$.

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



All parabolas share certain features.

Vertex – the lowest point (if the parabola opens up) or the highest point (if the parabola opens down).

The vertex of the basic parabola is V (0,0).

Axis of symmetry – the parabola is symmetric about the vertical line that runs through the vertex.

The axis of symmetry of the basic parabola is $x = 0$ (the y-axis).

***y*-intercept** – the point where the parabola intersects the *y*-axis.

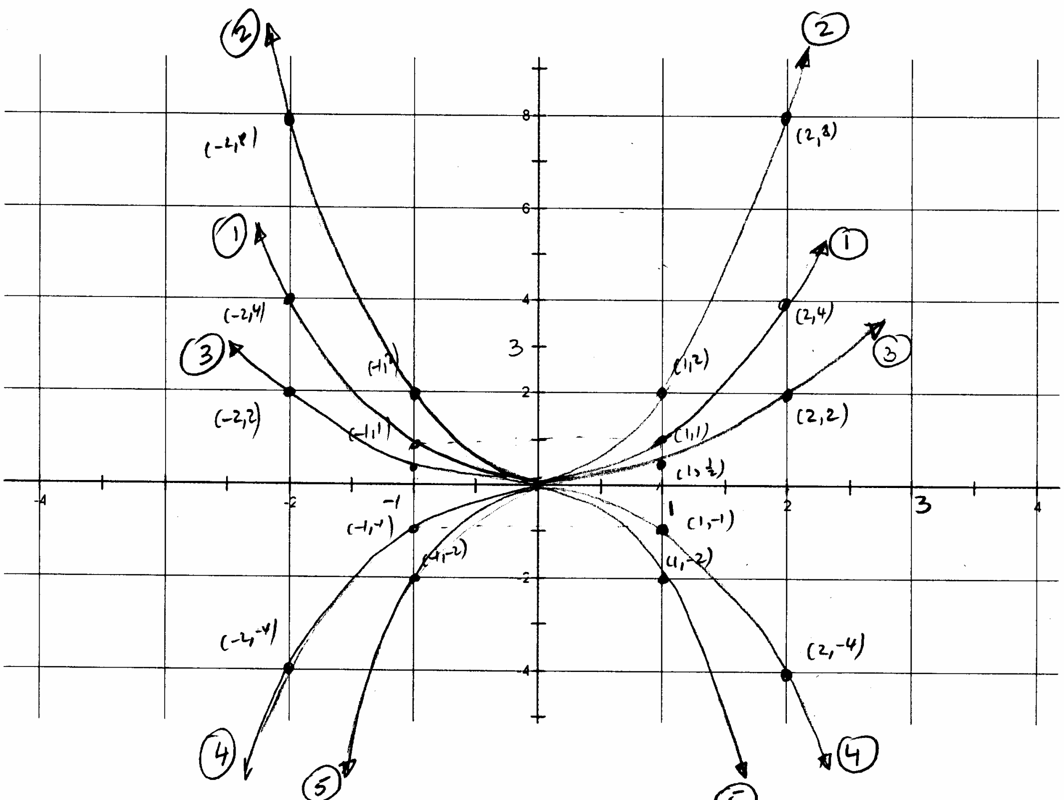
***x*-intercept(s)** – the point(s) where the parabola intersects the *x*-axis.

The x- and y-intercept of the basic parabola is (0,0).

Example #1 Graph the following parabolas on the same coordinate system:

1) $y = x^2$ 2) $y = 2x^2$ 3) $y = \frac{1}{2}x^2$ 4) $y = -x^2$ 5) $y = -2x^2$

Investigate the effect of the coefficient of x^2 on the graph.



$y = 2x^2$. its graph is obtained from the graph of $y = x^2$ stretched vertically by a factor of 2 (graph is narrower)

$y = \frac{1}{2}x^2$ is obtained from the graph of $y = \frac{1}{2}x^2$ by shrinking it vertically by a factor of 2 (graph is wider)

$y = -x^2$ is obtained by reflecting the graph of $y = x^2$ across the x-axis

What are the effects of the coefficient a of x^2 on the graph?

If $a > 0$, the parabola opens <u>upward</u>. vertex = minimum

If $a < 0$, the parabola opens <u>downward</u>. vertex = maximum

How to Graph a Parabola

Standard form: $y = ax^2 + bx + c \quad (a \neq 0)$

Note that if $a > 0$, the parabola opens upward, and if $a < 0$, the parabola opens downward.

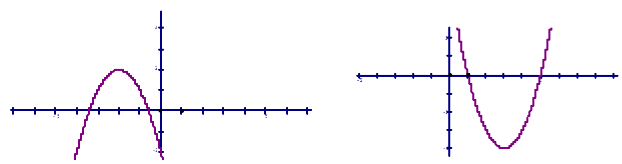
Vertex $V(x_v, y_v)$ $\boxed{x_v = \dfrac{-b}{2a}}$ To find y_v substitute the value of x_v in the equation and solve for y.

y-intercept To find the y-intercept make x=0 and solve for y.

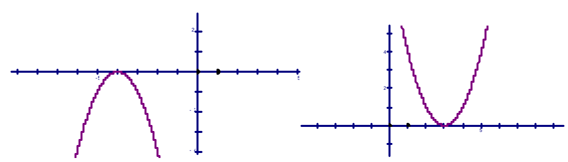
x-intercept(s) (if any) To find the x-intercept(s) make y=0 and solve for x.

How many x-intercepts?

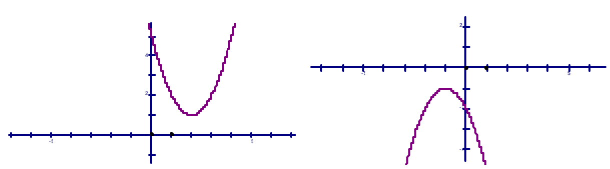
Two $x-\cap$ (when $\Delta > 0$)



One $x-\cap$ (when $\Delta = 0$)



No $x-\cap$ (when $\Delta < 0$)



Note: The parabola is symmetric about the vertical axis that passes through the vertex. If no x-intercept, use the symmetric of the y-intercept about the axis of symmetry to graph the parabola

The Vertex Form of a Parabola: $\boxed{y = a(x - x_v)^2 + y_v}$, where $V(x_v, y_v)$ is the vertex and a is the coefficient of x^2.

Exercise #1:

(a) Graph the following parabola: $y = x^2 + 3x + 2$. Give the domain and range.

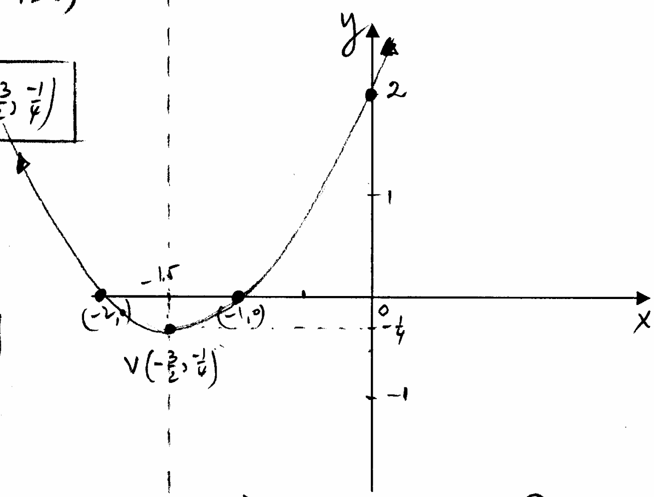
parabola opens up $(a=1>0)$

Vertex: $V(x_v, y_v)$

$x_v = \frac{-b}{2a} = \frac{-3}{2}$ $\quad \boxed{V\left(\frac{-3}{2}, \frac{-1}{4}\right)}$

$y_v = \left(\frac{-3}{2}\right)^2 + 3\left(\frac{-3}{2}\right) + 2 =$

$= \frac{9}{4} - \frac{9}{2} + 2 = \frac{-9}{4} + 2 = \frac{-1}{4}$



y-intercept:

$x = 0, y = 2$ $\quad \boxed{(0,2)}$

x-intercepts:

$y = 0$

$x^2 + 3x + 2 = 0$

$(x+1)(x+2) = 0$ $\quad \boxed{(-1,0) \text{ \& } (-2,0)}$

$x = -1$ OR $x = -2$

Domain: $x \in \mathbb{R}$

Range: $y \in [-\frac{1}{4}, \infty)$

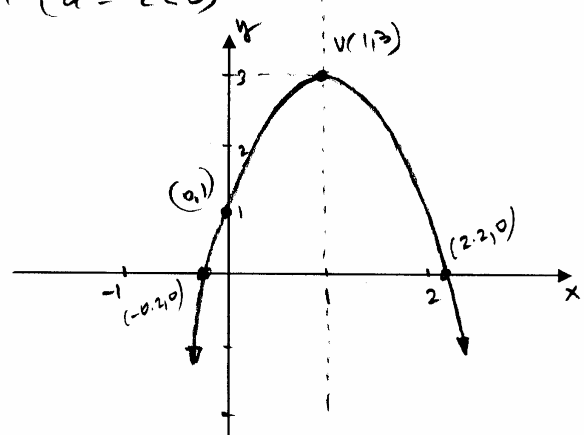
(b) Graph the following parabola: $y = -2x^2 + 4x + 1$. Give the domain and range.

parabola opens down $(a = -2 < 0)$

Vertex: $V(x_v, y_v)$

$x_v = \frac{-b}{2a} = \frac{-4}{2(-2)} = 1$ $\quad \boxed{V(1,3)}$

$y_v = -2(1)^2 + 4(1) + 1 = 3$



y-intercept:

$x = 0, y = 1$ $\quad \boxed{(0,1)}$

x-intercepts:

$y = 0$

$-2x^2 + 4x + 1 = 0$

$2x^2 - 4x - 1 = 0$

$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-1)}}{2(2)}$

$= \frac{4 \pm \sqrt{16+8}}{4} = \frac{4 \pm \sqrt{24}}{4} = \frac{4 \pm 2\sqrt{6}}{4}$

$= \frac{2 \pm \sqrt{6}}{2} <$ $x_1 \simeq 2.2$, $x_2 \simeq -0.2$ $\quad \boxed{(2.2,0) \text{ \& } (-0.2,0)}$

Domain: $x \in \mathbb{R}$

Range: $y \in (-\infty, 3]$

Exercise #2: Find the vertex of each parabola. Decide whether the vertex is a maximum or a minimum point. Give the domain and the range.

(a) $y = 2(x-3)^2 + 4$. Graph the function explaining how its graph is obtained from the graph of the basic parabola.

1st $y = x^2$

2nd $y = (x-3)^2$ horizontol shift to the right 3 units

3rd $y = 2(x-3)^2$ vertical stretch of the previous graph by a factor of 2

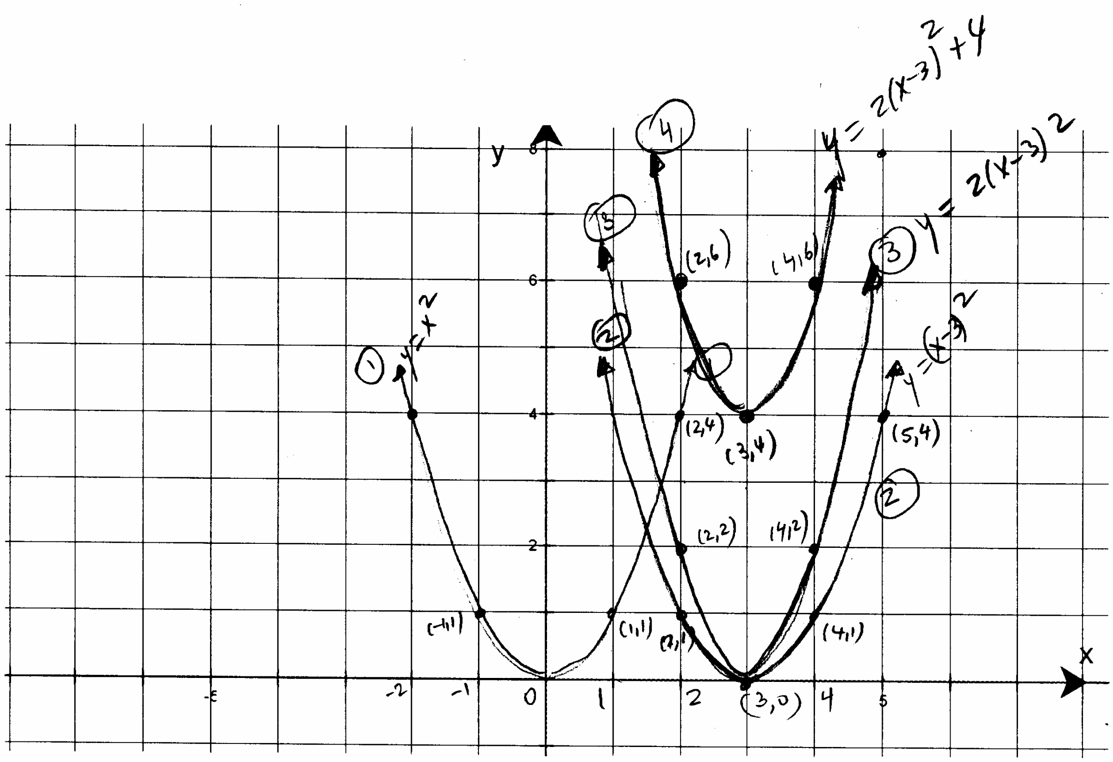
4th $y = 2(x-3)^2 + 4$ vertical shift up 4 units (of the previous graph)

$y = 2(x-3)^2 + 4$ - parabola opens up ($a = 2 > 0$)

$V(3,4)$ - minimum point (the equation is in vertex form $y = a(x-x_v)^2 + y_v$)

Domain: $x \in \mathbb{R}$

Range: $y \in [4, \infty)$



(b) $y = -3(x+3)^2 - 5$. Graph the function explaining how its graph is obtained from the graph of the basic parabola.

= parabala opens down $V(-3,-5)$

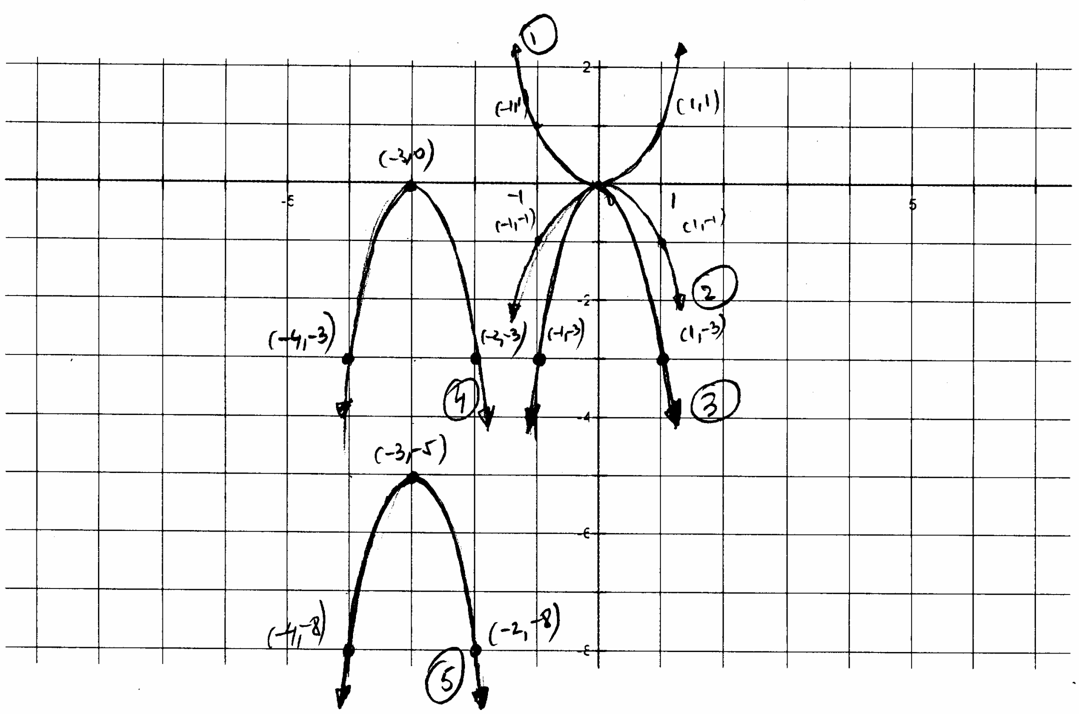
1st $y = x^2$

2nd $y = -x^2$ reflect the graph of $y = x^2$ about the x-axis

3rd $y = -3x^2$ stretch vertically (by a factor of 3) the graph of $y = -x^2$

4th $y = -3(x+3)^2$ shift left 3 units the graph of $y = -3x^2$

5th $y = -3(x+3)^2 - 5$ shift down 5 units the graph of $y = -3(x+3)^2$



(c) $y = 3x^2 + 4x + 2$. Graph this parabola by writing its equation in vertex form first (by completing the square on x).

I Completing the square on x

$y = 3(x^2 + \frac{4}{3}x) + 2$

$(\frac{1}{2}\text{coef } x)^2 = (\frac{1}{2}\cdot\frac{4}{3})^2 = (\frac{2}{3})^2 = \frac{4}{9}$

$y = 3(x^2 + \frac{4}{3}x + \frac{4}{9}) + 2 - \frac{4}{3}$

$\boxed{y = 3(x+\frac{2}{3})^2 + \frac{2}{3}}$ $V(\frac{-2}{3}, \frac{2}{3})$

II OR, find $V(x_v, y_v)$

$x_v = \frac{-b}{2a} = \frac{-4}{6} = \frac{-2}{3}$

$y_v = 3(\frac{-2}{3})^2 + 4(\frac{-2}{3}) + 2 = \frac{2}{3}$

$V(-\frac{2}{3}, \frac{2}{3})$

$y = a(x - x_v)^2 + y_v$

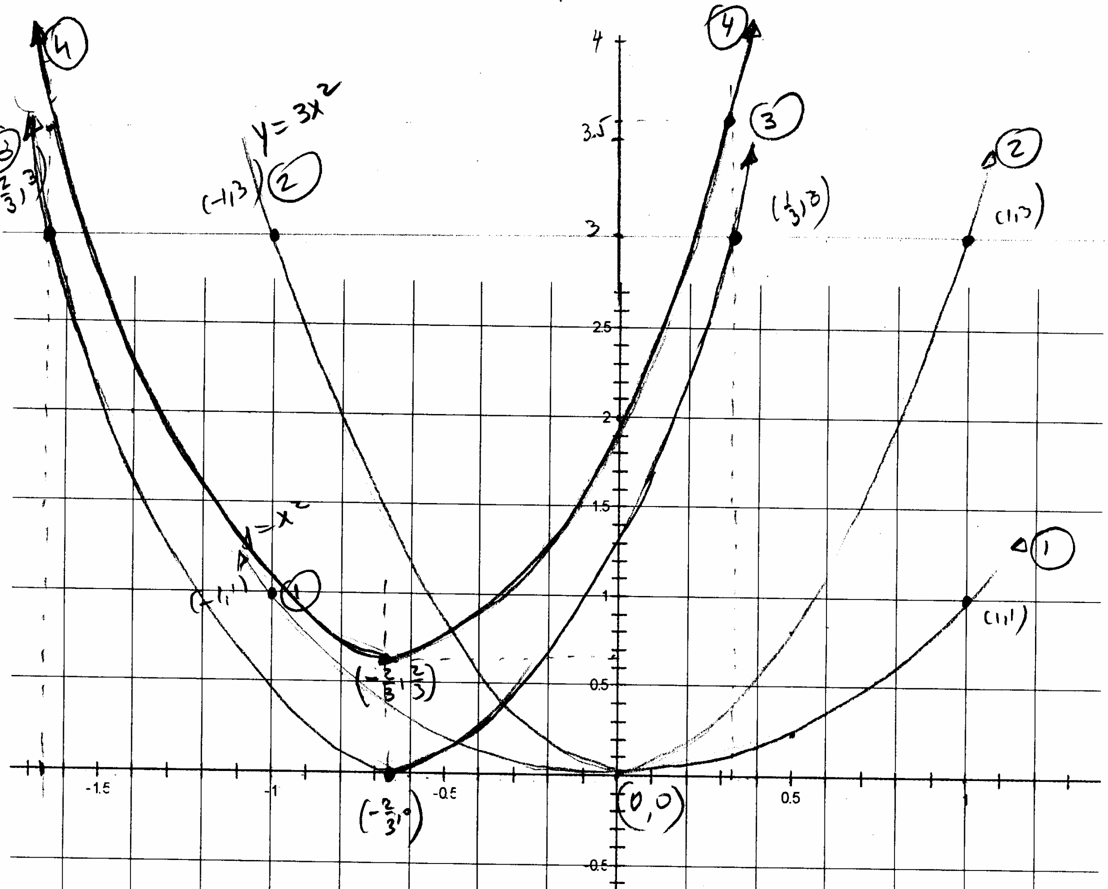
$y = 3(x + \frac{2}{3})^2 + \frac{2}{3}$

1st $y = x^2$

2nd $y = 3x^2$ vertical stretch by a factor of 3

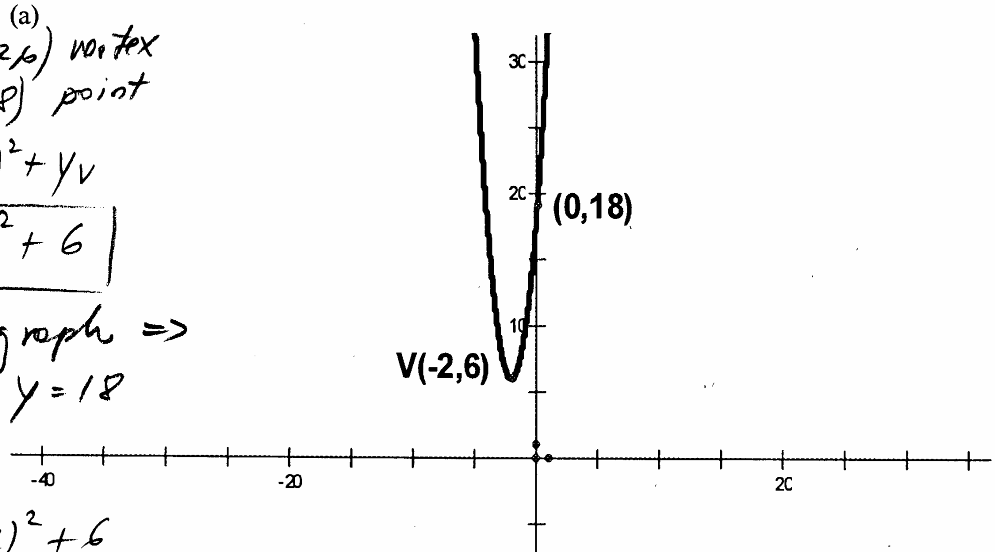
3rd $y = 3(x+\frac{2}{3})^2$ horizontal shift to the left $\frac{2}{3}$ (of the previous graph)

4th $y = 3(x+\frac{2}{3})^2 + \frac{2}{3}$ vertical shift up $\frac{2}{3}$ (of the previous graph)



Exercise #3: Write an equation for each graph. Give the domain and range.

(a)



Given: $V(-2,6)$ vertex

$(0,18)$ point

$y = a(x - x_V)^2 + y_V$

$$\boxed{y = a(x+2)^2 + 6}$$

$(0,18) \in$ graph $\Rightarrow$

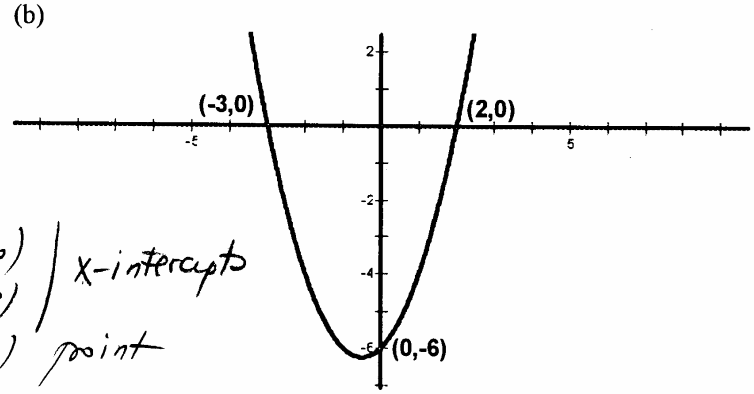
when $x = 0$, $y = 18$

$18 = a(0+2)^2 + 6$

$18 = 4a + 6 \Rightarrow a = 3$

Therefore, the parabola is $\boxed{y = 3(x+2)^2 + 6}$

(b)



Given $(-3,0)$, $(2,0)$ x-intercepts

$(0,-6)$ point

$$\boxed{y = a(x+3)(x-2)}$$

$(0,-6) \in$ graph $\Rightarrow$ when $x = 0$, $y = -6$

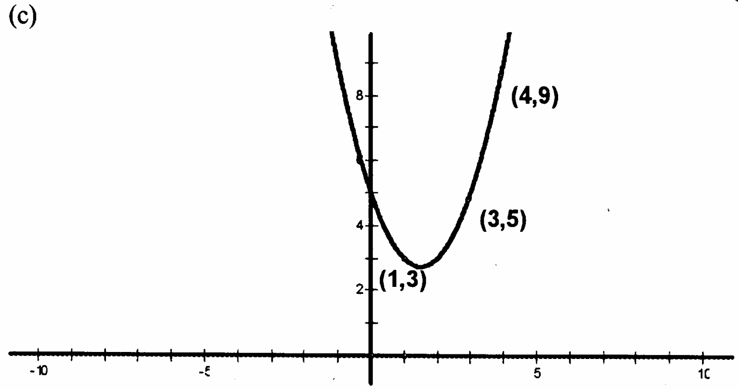
$-6 = a(0+3)(0-2)$

$-6 = -6a \Rightarrow a = 1$

Therefore, the parabola is $y = (x+3)(x-2)$

$$\boxed{y = x^2 + x - 6}$$

(c)



Given $\left\{ \begin{array}{l} (1,3) \\ (3,5) \\ (4,9) \end{array} \right.$ points

$y = ax^2 + bx + c \qquad a, b, c = ?$

$(1,3) \in$ parabola $\Rightarrow 3 = a + b + c$ ①

$(3,5) \in$ parabola $\Rightarrow 5 = 9a + 3b + c$ ②

$(4,9) \in$ parabola $\Rightarrow 9 = 16a + 4b + c$ ③

Solve the 3×3 system.

$$\begin{cases} a + b + c = 3 & ① \\ 9a + 3b + c = 5 & ② \\ 16a + 4b + c = 9 & ③ \end{cases}$$

$② - ① \Longrightarrow 8a + 2b = 2 \quad / \div 2$

$\boxed{4a + b = 1}$ ④

$③ - ② \Longrightarrow \boxed{7a + b = 4}$ ⑤

$$\begin{cases} 4a + b = 1 & ④ \\ 7a + b = 4 & ⑤ \end{cases}$$

$\ominus \quad -3a = -3 \Rightarrow a = 1$

$4a + b = 1$
$4 + b = 1$
$b = -3 \qquad b = -3$

$a + b + c = 3$
$1 - 3 + c = 3$
$c = 5 \qquad c = 5$

Therefore, the parabola is

$\boxed{y = x^2 - 3x + 5}$

Exercise #4 (3.1 - #47)

If air resistance is neglected, the height s (in feet) of an object propelled directly upward from a an initial height s_0 feet with initial velocity v_0 feet per second is

$$s(t) = -16t^2 + v_0 t + s_0,$$

where t is the number of seconds after the object is propelled.

s_0 = initial height
v_0 = initial velocity
t = time
$s(t)$ = height

A toy rocket is launched straight up from the top of a building 50 ft tall at an initial velocity of 200 ft per sec. $s_0 = 50$ $v_0 = 200$

a) Give the function that describes the height of the rocket in terms of t.

$s(t) = -16t^2 + 200t + 50$ parabola opens downward

b) Determine the time at which the rocket reaches its maximum height, and the maximum height in feet.

The maximum will occur at the vertex $V(t_v, s_v)$

$t_v = \frac{-b}{2a} = \frac{-200}{2(-16)} = 6.25$ sec.

$s_v = -16(6.25)^2 + 200(6.25) + 50$
$= 675$ ft

Therefore, the rocket will reach its maximum height after 6.25 sec and the max. height will be 675 ft

c) For what interval will the rocket be more than 300 feet above the ground level?

$t = ?$ so that $s(t) > 300$

$-16t^2 + 200t + 50 > 300$

$16t^2 - 200t - 50 + 300 < 0$

$16t^2 - 200t + 250 < 0 \quad | \div 2$

$8t^2 - 100t + 125 < 0$

parabola opens up

the t-n ax: $8t^2 - 100t + 125 = 0$

$t_{1/2} = \frac{100 \pm \sqrt{100^2 - 4 \cdot 8 \cdot 125}}{2 \cdot 8} \approx \frac{100 \pm 77.5}{16}$ $t_2 = 11.1$ sec, $t_1 = 1.4$ sec

$\boxed{t \in (1.4 \text{ sec}, 11.1 \text{ sec})}$

d) After how many seconds will it hit the ground?

$t = ?$ so that $s(t) = 0$

$-16t^2 + 200t + 50 = 0 \quad | (-1)$

$16t^2 - 200t - 50 = 0 \quad | \div 2$

$8t^2 - 100t - 25 = 0$

$t_{1/2} = \frac{100 \pm \sqrt{100^2 - 4 \cdot 8(-25)}}{2(8)} \approx \frac{100 \pm 104}{16}$ $\cancel{t_1 = -0.25}$, $t_2 = 12.75$ sec.

The rocket will hit the ground after 12.75 seconds.

Exercise #5 (3.1 - # 51)

Suppose that x represents one of two positive numbers whose sum is 30.

$x + \square = 30$

a) Represent the other of the two numbers in terms of x.

$30 - x$

b) What are the restrictions on x?

$0 < x < 30$

c) Determine a function f that represents the product of these two numbers.

$f(x) = x(30-x)$

$f(x) = -x^2 + 30x$

d) What are the two such numbers that yield the maximum product? What is their product?

$f(x) = -x^2 + 30x$ parabola opens down $\Rightarrow$ maximum occurs at the vertex

$V(x_v, y_v)$, where x = 1st number

$y = f(x)$ = the product of the two numbers.

$x_v = \frac{-b}{2a} = \frac{-30}{2(-1)} = 15$

$y_v = -15^2 + 30(15) = 225$

Therefore, the 1st # is 15

the 2nd # is also 15

and their product is 225.

e) For what two such numbers is the product equal to [redacted] 104 ?

$x = ?$ so that $f(x) = 104$

$-x^2 + 30x = 104$

$x^2 - 30x + 104 = 0$

$x_{1,2} = \frac{30 \pm \sqrt{30^2 - 4(104)}}{2(1)} = \frac{30 \pm \sqrt{900 - 416}}{2} = \frac{30 \pm \sqrt{484}}{2}$

$= \frac{30 \pm 22}{2}$ $\begin{cases} \frac{52}{2} = 26 \\ \frac{8}{2} = 4 \end{cases}$

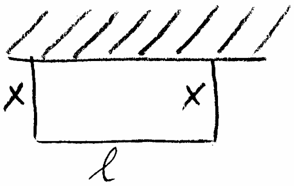
If the 1st number is 26, then the 2nd number is $30 - 26 = 4$

If the 2nd number is 4, then the 2nd number is $30 - 4 = 26$

The numbers must be 26 and 4 in order for the product to be 104

Exercise #6 (3.1 - # 53)

One campus has plans to construct a rectangular parking lot on land bordered on one side by a highway. There are 640 ft of fencing available to fence the other three sides. Let x represent the length of each of the two parallel sides of fencing.



a) Represent the length of the remaining side to be fenced in terms of x.

$2x + l = 640 \Rightarrow \boxed{l = 640 - 2x}$

b) What are the restrictions on x?

$0 < x < \frac{640}{2}$ $\quad \boxed{0 < x < 320}$

c) Determine a function A that represents the area of the parking lot in terms of x.

$A = l \cdot x \quad A(x) = x(640 - 2x) \quad \boxed{A(x) = -2x^2 + 640x}$

d) Determine the values of x that will give an area between 30,000 and 40,000 sq.ft.

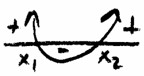
$x = ?$ so that $30{,}000 \underset{①}{\leq} -2x^2 + 640x \underset{②}{\leq} 40{,}000$

① $30{,}000 \leq -2x^2 + 640x$

AND

$2x^2 - 640x + 30{,}000 \leq 0$

$x^2 - 320x + 15{,}000 \leq 0$



$x_{1,2} = \dfrac{320 \pm \sqrt{(320)^2 - 4(15{,}000)}}{2(1)}$

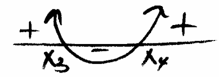
$\approx \dfrac{320 \pm 206}{2}$ < 57 ft, 263 ft

$x \in [57, 263]$ ①

② $-2x^2 + 640x \leq 40{,}000$

$2x^2 - 640x + 40{,}000 \geq 0$

$x^2 - 320x + 20{,}000 \geq 0$



$x_{3,4} = \dfrac{320 \pm \sqrt{(320)^2 - 4(20{,}000)}}{2(1)}$

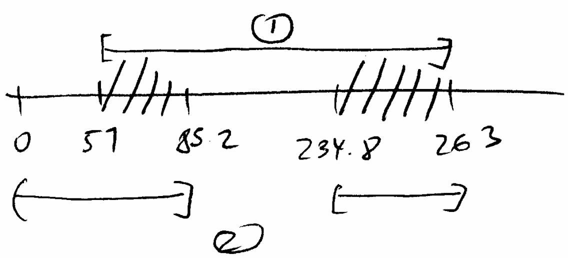
$\approx \dfrac{320 \pm 149.6}{2}$ < 85.2 ft, 234.8 ft

$x \in (0, 85.2] \cup [234.8, \infty)$ ②

Conditions ① and ② :

$x \in [57, 263] \cap \left((0, 85.2] \cup [234.8, \infty) \right)$

$\boxed{x \in [57, 85.2] \cup [234.8, 263]}$



e) What dimensions will give a maximum area, and what will this area be?

$A(x) = -2x^2 + 640x$ ← downward parabola

max. occurs at the vertex $V(x_v, A_v)$

$x_v = \frac{-b}{2a} = \frac{-640}{2(-2)} = 160$ ft

Max. area occurs if the width $x = 160$ ft

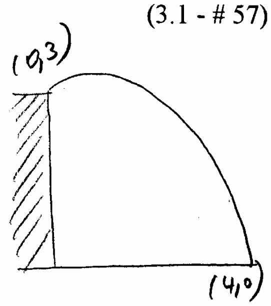
then the length $l = 640 - 2x$

$l = 640 - 2(160) = 320$ ft

The max. area is

$160(320) = 51{,}200$ ft^2

Exercise #7 (3.1 - # 57) A frog leaps from a stump 3 ft high and lands 4 ft from the base of the stump. We can consider the initial position of the frog to be (0,3) and its landing position to be at (4,0). It is determined that the height of the frog as a function of its horizontal distance x from the base of the stump is given by $h(x) = -0.5x^2 + 1.25x + 3$, where x and $h(x)$ are both in feet.



x = horizontal dist.
$h(x)$ = height

a) How high was the frog when its horizontal distance from the base of the stump was 2 ft?

$h = ?$ when $x = 2$

$h(2) = -0.5(2)^2 + 1.25(2) + 3$

$= 3.5$ ft

The frog was 3.5 ft high

b) At what horizontal distances from the base of the stump was the frog 3.25 ft above the ground?

$x = ?$ when $h = 3.25$ ft

$-0.5x^2 + 1.25x + 3 = 3.25$

$0.5x^2 - 1.25x + 0.25 = 0$

$$x_{1,2} = \frac{1.25 \pm \sqrt{(1.25)^2 - 4(0.5)(0.25)}}{2(0.5)} = \frac{1.25 \pm \sqrt{1.0625}}{1} \approx 1.25 \pm 1.03 \begin{cases} 2.28 \text{ ft} \\ 0.22 \text{ ft} \end{cases}$$

The frog was 3.25 ft above the ground when it was approximately 0.22 ft and 2.28 ft from the base of the stump

c) At what horizontal distance from the base of the stump did the frog reach its highest point?

$h(x) = -0.5x^2 + 1.25x + 3$ downward parabola

maximum occurs at the vertex $V(x_v, h_v)$

$x_v = \frac{-b}{2a} = \frac{-1.25}{2(-0.5)} = 1.25$ ft

So, the frog reached its highest point when it was 1.25 ft from the base of the stump.

d) What was the maximum height reached by the frog?

$h_{max} = h_v = -0.5(1.25)^2 + 1.25(1.25) + 3$

≈ 3.78 ft

The max. height reached was 3.78 ft.

Exercise #8 (3.1 - #71) Find a value of c so that $y = x^2 - 10x + c$ has exactly one x-intercept.

x-n: $y = 0$

$x^2 - 10x + c = 0$

one x-n iff $x^2 - 10x + c = 0$ has one solution

iff $\Delta = b^2 - 4ac = 0$

$\Delta = 100 - 4c = 0$

$100 = 4c \iff$ $c = 25$

Exercise #9 (3.1 - #75) Find the largest possible value of y if $y = -(x-2)^2 + 9$. Then find the following:

a) the largest possible value of $\sqrt{-(x-2)^2 + 9}$

$y = -(x-2)^2 + 9$

parabola opens downward, therefore the max. occurs at the vertex

$V(2, 9)$ $\boxed{y_{max} = 9}$

$\sqrt{-(x-2)^2+9}$ increases if $-(x-2)^2+9$ increases

decreases if $-(x-2)^2+9$ decreases

$\sqrt{-(x-2)^2+9}$ = maximum if and only if $-(x-2)^2+9$ = maximum

$\boxed{\text{max. of } \sqrt{-(x-2)^2+9} = \sqrt{9} = 3}$

b) the smallest possible positive value of $\dfrac{1}{-(x-2)^2 + 9}$

when $y = -(x-2)^2+9$ is maximum,

$\frac{1}{y}$ is minimum

$\boxed{\text{The smallest value of } \dfrac{1}{-(x-2)^2+9} = \dfrac{1}{9}}$