3.2 Synthetic Division
3.3 Zeros of Polynomial Equations

In these sections we will study polynomials algebraically. Most of our work will be concerned with finding the solutions of polynomial equations of any degree – that is, equations of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0 \qquad (1)$$

Definition A **root** or **solution** of equation (1) is a number k that when substituted for x leads to a true statement. Thus, k is a root of equation (1) provided $f(k) = 0$.
We also refer to the number c in this case as **a zero of the function *f*.**

Exercise #1 Checking for a zero or root.

a) Is -1 a zero of $P(x) = -x^3 + x^2 - x + 1$?

$x=-1$ is a zero if and only if $P(-1)=0$

$P(-1) = -(-1)^3 + (-1)^2 - (-1) + 1 = 1+1+1+1 \neq 0$

Therefore, -1 is NOT a zero.

b) Is $x = \frac{1}{2}$ a root of the equation $2x^2 - 3x + 1 = 0$?

$x=\frac{1}{2}$ is a root if and only if $2\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 1 = 0$

$2 \cdot \frac{1}{4} - \frac{3}{2} + 1 = 0$

$\frac{1}{2} - \frac{3}{2} + 1 = 0$ TRUE

Therefore, $x=\frac{1}{2}$ is a root of the equation.

Note: If a root is repeated *n* times, we call it a **root of multiplicity *n***

Exercise #2 a) State the multiplicity of each root of the equation: $x^2(x+1)^3(x-1) = 0$ =>

or $x^2=0 \Rightarrow x=0$ root of multiplicity 2

$(x+1)^3=0 \Rightarrow x=-1$ root of multiplicity 3

or $x-1=0 \Rightarrow x=1$ root of multiplicity 1

(3.3 - #46) b) Find all zeros and their multiplicities: $f(x) = 5x^2\left(x+1-\sqrt{2}\right)(2x+5)$

$5x^2(x+1-\sqrt{2})(2x+5)=0 \Rightarrow$

$x^2=0 \Rightarrow x=0$ root of multiplicity 2

or $x+1-\sqrt{2}=0 \Rightarrow x=-1+\sqrt{2}$ root of multiplicity 1

or $2x+5=0 \Rightarrow x=-\frac{5}{2}$ root of multiplicity 1

(3.3 - #48) c) Find all zeros and their multiplicities: $f(x) = (7x-2)^3(x^2+9)^2$

$(7x-2)^3(x^2+9)^2=0 \Rightarrow$

$(7x-2)^3=0 \Rightarrow 7x-2=0 \Rightarrow x=\frac{2}{7}$ root of multiplicity 3

or $(x^2+9)^2=0 \Rightarrow x^2+9=0$

$x^2=-9$

$x=\pm 3i$ $\Rightarrow$ $x=3i$ root of multiplicity 2

$x=-3i$ root of multiplicity 2

Division of Polynomials

The process of long division for polynomials follows the same four-step cycle used in ordinary long division of numbers: divide, multiply, subtract, bring down.
Notice that in setting up the division, we write both the dividend and divisor in decreasing powers of x.

Example #1 Divide $5x^3 - 6x^2 - 28x - 2$ by $x+2$.
(3.2 – Example 1)

$$
\begin{array}{r|l}
 & 5x^2 - 16x + 4 \\
\hline
x+2 & 5x^3 - 6x^2 - 28x - 2 \\
 & -5x^3 - 10x^2 \\
\hline
 & -16x^2 - 28x - 2 \\
 & +16x^2 + 32x \\
\hline
 & 4x - 2 \\
 & -4x - 8 \\
\hline
 & \boxed{-10} \text{ the Remainder}
\end{array}
\qquad
\frac{5x^3}{x} = 5x^2 \quad \frac{-16x^2}{x} = -16x \quad \frac{4x}{x} = 4
$$

The result of the division can be written as: $\dfrac{5x^3-6x^2-28x-2}{x+2} = 5x^2 - 16x + 4 + \dfrac{-10}{x+2}$

or

$$\underbrace{5x^3-6x^2-28x-2}_{\text{DIVIDEND } f(x)} = \underbrace{(x+2)}_{\text{DIVISOR } \cdot g(x)}\underbrace{(5x^2-16x+4)}_{\text{QUOTIENT } q(x)} + \underbrace{(-10)}_{\text{REMAINDER } r(x)}$$

Note 1) Second equation is valid for all real numbers x, whereas first equation carries implicit restrictions that x my not equal -2. For this reason, we often prefer to write our results in the form of the second equation.

2) The degree of the remainder is less than the degree of the divisor. This is very similar to the situation with ordinary division of positive integers, where the remainder is always less than the divisor.

The Division Algorithm
(3.2)

Let $f(x)$ and $g(x)$ be polynomials with $g(x)$ of lower degree than $f(x)$ and assume that $g(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$ such that

$$\boxed{f(x) = g(x) \cdot q(x) + r(x)}$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$. The polynomials $f(x)$ and $g(x)$ are called the **dividend** and **divisor**, respectively, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

When $r(x) = 0$, we have $f(x) = g(x) \cdot q(x)$ and we say that $g(x)$ and $q(x)$ are **factors** of $f(x)$.

Exercise #3 Using long division to find a quotient and a remainder.

Divide x^3+2x^2-4 by $x-3$.

$$
\begin{array}{r|l}
 & x^2+5x+15 \\
\hline
x-3 & x^3+2x^2+0x-4 \\
 & -x^3+3x^2 \\
\hline
 & 5x^2+0x-4 \\
 & -5x^2+15x \\
\hline
 & 15x-4 \\
 & -15x+45 \\
\hline
 & \boxed{41} \text{ Remainder}
\end{array}
$$

$\frac{x^3}{x}=x^2$

$\frac{5x^2}{x}=5x$

$\frac{15x}{x}=15$

$x^3+2x^2-4=(x-3)(x^2+5x+15)+41$

$x^2+5x+15=$ the quotient

$41=$ the remainder.

Exercise #4 Use long division to find whether $x+3$ is a factor of $x^2-2x-15$.

$x+3$ is a factor of $x^2-2x-15$ iff when dividing $x^2-2x-15$ by $x+3$ the remainder is 0.

$$
\begin{array}{r|l}
 & x-5 \\
\hline
x+3 & x^2-2x-15 \\
 & -x^2-3x \\
\hline
 & -5x-15 \\
 & +5x+15 \\
\hline
 & \boxed{0} \text{ R}
\end{array}
$$

Therefore, $x+3$ is a factor of $x^2-2x-15$

$x+3 \mid x^2-2x-15$

Synthetic Division

- Synthetic division is a quick method of dividing polynomials.
- It can be used when the divisor is of the form $x-k$.
- In the synthetic division we write down only the essential parts of the long division table (the coefficients).

Example #2 Divide, using synthetic division, $x^2-2x-15$ by $x+3$.

$$
\begin{array}{r|rrr}
 & 1 & -2 & -15 \\
\hline
-3 & \underbrace{1 \quad -5}_{\text{the coefficients of the quotient}} & & \boxed{0}
\end{array}
$$

Remainder

$x+3=x-(-3)$

$k=-3$

$x^2-2x-15=(x+3)(x-5)$

Note that the degree of the quotient is one less than the degree of the dividend.

Exercise #4 Use synthetic division to perform the following divisions:
(3.2 - #2, 11)

a) $\dfrac{x^3+4x^2-5x+44}{x+6}$

$x+6 = x-(-6)$

$$\begin{array}{r|rrrr} & 1 & 4 & -5 & 44 \\ \hline -6 & \underbrace{1 \quad -2 \quad 7}_{\text{QUOTIENT}} & & & \boxed{2}\ R \end{array}$$

$$\frac{x^3+4x^2-5x+44}{x+6} = x^2-2x+7+\frac{2}{x+6}$$

If $f(x)=x^3+4x^2-5x+44$, evaluate $f(-6)$. What do you observe?

$f(-6)=(-6)^3+4(-6)^2-5(-6)+44$
$f(-6)=2$

We see that $f(-6)=2=$ the remainder when dividing $f(x)$ by $x+6$

b) $\dfrac{\frac{1}{3}x^3-\frac{2}{9}x^2+\frac{1}{27}x+1}{x-\frac{1}{3}}$

$$\begin{array}{r|rrrr} & \frac{1}{3} & -\frac{2}{9} & \frac{1}{27} & 1 \\ \hline \frac{1}{3} & \underbrace{\frac{1}{3} \quad -\frac{1}{9} \quad 0}_{\text{Quotient}} & & & \underset{R}{\boxed{1}} \end{array}$$

$$\frac{\frac{1}{3}x^3-\frac{2}{9}x^2+\frac{1}{27}x+1}{x-\frac{1}{3}} = \frac{1}{3}x^2-\frac{1}{9}x+\frac{1}{x-\frac{1}{3}}$$

If $f(x)=\frac{1}{3}x^3-\frac{2}{9}x^2+\frac{1}{27}x+1$, evaluate $f\left(\frac{1}{3}\right)$. What do you observe?

$f(\tfrac{1}{3})=\tfrac{1}{3}(\tfrac{1}{3})^3-\tfrac{2}{9}(\tfrac{1}{3})^2+\tfrac{1}{27}\cdot\tfrac{1}{3}+1=1$

We see that $f(\frac{1}{3})=1=$ the remainder when dividing $f(x)$ by $x-\frac{1}{3}$.

The Remainder Theorem (3.2)

When we divide a polynomial $f(x)$ by $x-k$, the remainder is $f(k)$.

Proof

From the Division Algorithm $\Rightarrow f(x)=(x-k)q(x)+r(x)$
where degree $r(x) <$ degree $(x-k)$
but degree$(x-k)=1$ $\Rightarrow$ degree $r(x)=0$
$\Rightarrow r(x)=$ constant $=c$

$$\underline{f(x)=(x-k)q(x)+c}$$

$f(k)=(k-k)q(x)+c$
$f(k)=0+c$
$\boxed{f(k)=c}$

Therefore, when dividing $f(x)$ by $x-k$, the remainder is $f(k)$

Exercise #5 (3.2 - #27, 35) Using the remainder theorem to evaluate a function and check for a factor.

a) Let $f(x) = x^2 + 5x + 6$.

i) Evaluate $f(-2)$.

ii) Is $x+2$ a factor of $f(x) = x^2 + 5x + 6$?

(i) $f(-2)$ = the remainder when dividing $f(x)$ by $x+2$

synthetic division

$$\begin{array}{r|rrr} & 1 & 5 & 6 \\ -2 & 1 & 3 & \boxed{0} \end{array} \; R \quad \Rightarrow \quad \boxed{f(-2)=0}$$

(ii) Yes, $x+2$ is a factor of $f(x)$ because, when dividing $f(x)$ by $x+2$ the remainder is 0.

So, $x+2 \mid f(x)$

$f(x) = (x+2)(x+3)$

b) Let $f(x) = 6x^4 + x^3 - 8x^2 + 5x + 6$.

i) Evaluate $f\left(\frac{1}{2}\right)$.

ii) Is $x - \frac{1}{2}$ a factor of $f(x) = 6x^4 + x^3 - 8x^2 + 5x + 6$?

(i)
$$\begin{array}{r|rrrrr} & 6 & 1 & -8 & 5 & 6 \\ \frac{1}{2} & 6 & 4 & -6 & 2 & \boxed{7} \end{array}$$

So, from the Remainder th. $\Rightarrow$ $\boxed{f(\tfrac{1}{2}) = 7}$

(ii) $x - \frac{1}{2}$ is not a factor of $f(x)$ because the remainder $\neq 0$

$x - \frac{1}{2} \nmid f(x)$

The Factor Theorem (3.3)

The polynomial $x-k$ is a factor of the polynomial $f(x)$ if and only if $f(k)=0$.

Exercise #6 Let $f(x) = 2x^3 - 4x^2 + 2x - 1$.

a) What is the remainder when dividing the given polynomial by $x-2$? In how many ways can you find the remainder? Which method is the easiest one?

The remainder could be found by
- long division
  OR
- synthetic division
  OR
- the remainder th.

the easy methods :)

$$\begin{array}{r|rrrr} & 2 & -4 & 2 & -1 \\ 2 & 2 & 0 & 2 & \boxed{3} \end{array}$$

$\boxed{R = 3}$

b) Is $x-2$ a factor of $f(x)$? No, because $R \neq 0$ (from (a))

$x-2 \nmid f(x)$

c) Is $x-1$ a factor of $f(x)$?

Factor th. $\Rightarrow$ $x-1$ is a factor of $f(x)$ iff $f(1)=0$

$f(1) = 2-4+2-1 = -1 \neq 0$ Therefore, $x-1$ is not a factor of $f(x)$ $\quad x-1 \nmid f(x)$

Exercise #7 <u>Factoring a polynomial given a zero.</u>

a) Let $f(x) = x^3 - 7x + 6$. Show that $f(1) = 0$ and use this fact to factor $f(x)$ completely.

$f(1) = 1-7+6 = 0$

From the Factor th $\Rightarrow$ $x-1 \mid f(x)$

$$\begin{array}{c|cccc} & 1 & 0 & -7 & 6 \\ \hline 1 & \underbrace{1 \quad 1 \quad -6}_{\text{Quotient}} & & & \boxed{0}\ R \end{array}$$

$f(x) = (x-1)(x^2+x-6)$

$\boxed{f(x) = (x-1)(x+3)(x-2)}$

(3.3 - #19) b) Let $f(x) = 6x^3 + 13x^2 - 14x + 3$. Show that -3 is a zero and use this fact to factor $f(x)$ completely.

$x=-3$ is a zero iff $f(-3)=0$

$f(-3) = 6(-3)^3 + 13(-3)^2 - 14(-3) + 3 = 0$

OR using synthetic division, we show $f(-3) = R = 0$

$$\begin{array}{c|cccc} x & 6 & 13 & -14 & 3 \\ \hline -3 & \underbrace{6 \quad -5 \quad 1}_{\text{Quotient}} & & & \boxed{0}\ R \end{array}$$

$f(x) = (x+3)(6x^2-5x+1)$

$\boxed{f(x) = (x+3)(3x-1)(2x-1)}$

(3.3 - #28) c) $f(x) = 2x^4 + x^3 - 9x^2 - 13x - 5$. Knowing that -1 is a root of multiplicity 3, factor $f(x)$ into linear factors.

$x=-1$ root of multiplicity 3 iff $(x+1)^3 \mid f(x)$

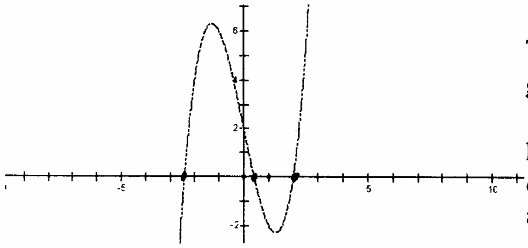
$$\begin{array}{c|ccccc} & 2 & 1 & -9 & -13 & -5 \\ \hline -1 & 2 & -1 & -8 & -5 & \boxed{0} \\ \hline -1 & 2 & -3 & -5 & \boxed{0} & \\ \hline -1 & 2 & -5 & \boxed{0} & & \end{array}$$

$f(x) = (x+1)(2x^3 - x - 8x - 5)$

$f(x) = (x+1)^2(2x^2 - 3x - 5)$

$\boxed{f(x) = (x+1)^3(2x-5)}$

Exercise #8 Applying the factor theorem in solving an equation.



The figure shows the graph of $y = x^3 - 5x + 2$. As indicated by the graph, the equation

$$x^3 - 5x + 2 = 0$$

has (at least) three roots, one of which either is equal to 2 or is very close to 2. Confirm that $x = 2$ is indeed a root of the above equation, and then solve the equation.

$x=2$ is a root iff $2^3-5(2)+2=0$
$8-10+2=0$ true $\Rightarrow$ $\underline{x=2 \text{ is a root}}$

$x=2$ is a root $\Rightarrow$ $x-2 \mid x^3-5x+2$

	1	0	-5	2
2	1	2	-1	$\boxed{0}$

$x^3-5x+2=(x-2)(x^2+2x-1)=0$

$x^2+2x-1=0$

$x=\dfrac{-2\pm\sqrt{4+4\cdot1}}{2}=\dfrac{-2\pm2\sqrt{2}}{2}=-1\pm\sqrt{2}$

The solutions of the equation $x^3-5x+2=0$ are $\boxed{\{2, -1\pm\sqrt{2}\}}$

The Conjugate Zeros Theorem
(3.3)

> If $f(x)$ is a polynomial function with real coefficients and if $a+bi$ is a zero of $f(x)$, then its conjugate $a-bi$ is also a zero of $f(x)$.

Exercise #9 For each polynomial, one zero is given. Find all the others.
(3.3 - #31, 32)

a) $f(x) = x^3 - 7x^2 + 17x - 15$; $2-i$

if $x=2-i$ is a root, then
$x=2+i$ is also a root.

Factor th.
$\Rightarrow x-(2-i) \mid f(x)$
$\Rightarrow x-(2+i) \mid f(x)$

	1	-7	17	-15
$2-i$	1	$-5-i$	$6+3i$	$\boxed{0}$
$2+i$	1	-3	$\boxed{0}$	

$-10-2i+5i-1$
$-11+3i+17$
$12+6i-6i+3$
$15-15$
$2+i-5-i$
$-6-3i+6+3i$

$f(x)=(x-(2-i))(x^2+(-5-i)x+6+3i)$

$f(x)=(x-(2-i))(x-(2+i))(x-3)$

The zeros are:

$$\boxed{\begin{aligned} x &= 2-i \\ x &= 2+i \\ x &= 3 \end{aligned}}$$

b) $f(x)=4x^3+6x^2-2x-1;\ \frac{1}{2}$

$\frac{1}{2}$ = zero $\Rightarrow f(\frac{1}{2})=0 \Rightarrow x-\frac{1}{2} \mid f(x)$
(Factor th)

$$\begin{array}{c|cccc} & 4 & 6 & -2 & -1 \\ \hline \frac{1}{2} & 4 & 8 & 2 & \boxed{0} \end{array}$$

$f(x)=(x-\frac{1}{2})(4x^2+8x+2)$

$x_1=\frac{1}{2}$ or $4x^2+8x+2=0 \quad |\div 2$

$2x^2+4x+1=0$

$x_{2,3}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{8}}{4}=\frac{-4\pm2\sqrt{2}}{4}=\frac{-2\pm\sqrt{2}}{2}$

$x_{2,3}=\frac{-2\pm\sqrt{2}}{2}$

The Fundamental Theorem of Algebra
(3.3)

> Every polynomial equation of the form
> $$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0=0 \quad (n\geq1, a_n\neq0)$$
> has at least one root within the complex number system. (This root may be a real number).

What does it really say?
that there is k_1 = root ; $f(k_1)=0 \xRightarrow{\text{Factor th}} f(x)=(x-k_1)q_1(x)$
there is k_2 = root for q_1, $q_1(k_2)=0 \Rightarrow q_1(x)=(x-k_2)q_2(x)$
so $f(x)=(x-k_1)(x-k_2)q_2(x)$
⋮

Assuming $f(x)$ has degree n and repeating this process n times gives:
$f(x)=a_n(x-k_1)(x-k_2)\ldots(x-k_n)$.
Each of these factors leads to a zero of $f(x)$.
So $f(x)$ has n zeros.

The Linear Factors Theorem

> Every polynomial of degree n can be expressed as a product of n linear factors.
> $$f(x)=a_n(x-x_1)(x-x_2)\ldots(x-x_n),$$
> where a_n is the leading coefficient and x_i are the roots of the polynomial.

Theorem

> Every polynomial of degree $n\geq1$ has exactly n roots, where a root of multiplicity k is counted k times.

Exercise #10 Write each polynomial as a product of linear factors.

a) $f(x) = 3x^2 - 5x - 2$

Method I

Factoring

$\boxed{f(x) = (3x+1)(x-2)}$

Method II - Using the Linear Factors Theorem

We need to find all the zeros:

$f(x) = 0$

$3x^2 - 5x - 2 = 0$

$x = \frac{5 \pm \sqrt{25 - 4(3)(-2)}}{6} = \frac{5 \pm 7}{6}$

$x_1 = 2$

$x_2 = -\frac{1}{3}$

$\boxed{f(x) = 3(x-2)(x+\frac{1}{3})}$

b) $f(x) = x^2 - 5$

$x^2 - 5 = 0$

$x^2 = 5$

$x = \pm\sqrt{5}$ the zeros

$\boxed{f(x) = (x - \sqrt{5})(x + \sqrt{5})}$

c) $f(x) = x^2 - 4x + 5$

$x^2 - 4x + 5 = 0$

$x_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2}$

$= \frac{4 \pm 2i}{2}$ $\quad x_1 = 2 + i$, $\quad x_2 = 2 - i$

$f(x) = (x - (2+i))(x - (2-i))$

$\boxed{f(x) = (x - 2 - i)(x - 2 + i)}$

Exercise #11 Find a quadratic function whose zeros are 3 and 5 and whose graph passes through (2, -9).

If $x_1 = 3$

$x_2 = 5$

then $f(x) = a(x - x_1)(x - x_2)$

(The Linear Factors Theorem)

$f(x) = a(x-3)(x-5)$

$(2, -9) \in$ graph : when $x = 2$, $y = -9$

$-9 = a(-1)(-3)$

$-9 = 3a$

$a = -3$

$\boxed{f(x) = -3(x-3)(x-5)}$

$\boxed{f(x) = -3x^2 + 24x - 45}$

Exercise #12 Finding polynomial equations satisfying given conditions.

In each case, find a polynomial equation $f(x) = 0$ satisfying the given conditions. If there is no such equation, say so.

a) The numbers -1, 4, and 5 are roots.

$x = -1$ root $\overset{\text{Factor}}{\underset{\text{theorem}}{\Longleftrightarrow}}$ $x + 1 \mid f(x)$

$x = 4$ root $\Rightarrow$ $x - 4 \mid f(x)$

$x = 5$ root $\Rightarrow$ $x - 5 \mid f(x)$

$f(x) = (x+1)(x-4)(x-5) = 0$

is the simplest polynomial eq.

$f(x) = x^3 - 8x^2 + 11x + 20 = 0$

(3.3 - #49) b) Find a polynomial function of degree 3 having the numbers -3, 1, and 4 as roots and satisfying $f(2)=30$.

$x=-3$ root $\xrightarrow{\text{factor th}}$ $x+3 \mid f(x)$

$x=1$ root $\Rightarrow$ $x-1 \mid f(x)$

$x=4$ root $\Rightarrow$ $x-4 \mid f(x)$

degree $f(x)=3 \Rightarrow x+3$, $x-1$, and $x-4$ are the only factors

so, $f(x)=a(x+3)(x-1)(x-4)$

$\Rightarrow f(2)=30 \Rightarrow$

$30=a(5)(1)(-2)$

$30=-10a \Rightarrow a=-3$

$\boxed{f(x)=-3(x+3)(x-1)(x-4)}$

c) A factor of $f(x)$ is $x-3$, and -4 is a root of multiplicity 2.

$x-3 \mid f(x)$

$x=-4$ root of multiplicity 2 $\Rightarrow$

$\Rightarrow (x+4)^2 \mid f(x)$

$\Rightarrow$ The simplest polynomial equation is

$\boxed{f(x)=(x-3)(x+4)^2=0}$

(3.3 - #53) d) Find a polynomial function of degree 3 having the number -3 as a zero of multiplicity 3 and satisfying the condition $f(3)=36$.

$x=-3$ root of multiplicity 3 $\Rightarrow (x+3)^3 \mid f(x)$

degree $f(x)=3 \Rightarrow$ that's the only factor

$f(x)=a(x+3)^3$

$f(3)=36 \Rightarrow$

$36=a(6)^3 \Rightarrow a=\frac{1}{6}$ $\Rightarrow$ $\boxed{f(x)=\frac{1}{6}(x+3)^3}$

<u>**Exercise # 13**</u> Find a polynomial $f(x)$ with <u>leading coefficient 1</u> such that the equation $f(x)=0$ has only the following root: 3 having multiplicity 2, -2 having multiplicity 1 and 0 having multiplicity 2. What is the degree of this polynomial?

$x=3$ root of mult. 2 $\Rightarrow (x-3)^2 \mid f(x)$

$x=-2$ root of mult. 1 $\Rightarrow x+2 \mid f(x)$

$x=0$ root of mult. 2 $\Rightarrow x^2 \mid f(x)$

$\Rightarrow$ degree $f(x)=2+1+2$

degree $f(x)=5$

$\boxed{f(x)=x^2(x+2)(x-3)^2}$

Exercise #14 (3.3 - #57, 68) Find a polynomial function of least degree having only real coefficients with zeros as given. What is the degree of each polynomial?

a) 2 and $1+i$.

$x=2$ root $\Rightarrow$ $x-2 \mid f(x)$

$x=1+i$ root $\Rightarrow$ $x-(1+i) \mid f(x)$

Also,

$x=1-i$ root $\Rightarrow$ $x-(1-i) \mid f(x)$

degree $f(x)=3$

$f(x)=(x-2)(x-1-i)(x-1+i)$

$\Rightarrow f(x)=(x-2)((x-1)-i)((x-1)+i)$

$f(x)=(x-2)((x-1)^2-i^2)$

$f(x)=(x-2)(x^2-2x+1+1)$

$f(x)=(x-2)(x^2-2x+2)$

$\boxed{f(x)=x^3-4x^2+6x-4}$

b) $5+i$ and $4-i$.

$x=5+i$ root $\Rightarrow$ $x=5-i$ root $\Rightarrow$ $x-(5+i) \mid f(x)$, $x-(5-i) \mid f(x)$

$x=4-i$ root $\Rightarrow$ $x=4+i$ root $\Rightarrow$ $x-(4-i) \mid f(x)$, $x-(4+i) \mid f(x)$

$f(x)=(x-5-i)(x-5+i)(x-4+i)(x-4-i)$

$f(x)=((x-5)-i)((x-5)+i)((x-4)+i)((x-4)-i)$

$\Rightarrow f(x)=[(x-5)^2-i^2][(x-4)^2-i^2]$

$\boxed{f(x)=(x^2-10x+26)(x^2-8x+17)}$

Finding all the rational zeros of a polynomial

The Factor Theorem tells us that finding the zeros of a polynomial is really the same thing as factoring it into linear factors. We now study a method for finding all the rational zeros of a polynomial.

Example #3 Consider the polynomial

$$f(x)=(x-2)(x-3)(x+4)$$ Factored form

$$= x^3-x^2-14x+24$$ Expanded form.

What are the zeros of $f(x)$? $2, 3, -4$

What relationship exists between the zeros and the constant term of the polynomial?

2, 3, and -4 are factors of 24.

The next theorem generalizes this observation.

The Rational Zeros Theorem

(3.3)

If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ has integer coefficients, then every **rational zero** of $f(x)$ is of the form $\frac{p}{q}$ where p is a factor of the constant coefficient a_0; q is a factor of the leading coefficient a_n.

Note: The Rational Zeros Theorem gives only POSSIBLE rational zeros. It does not tell us whether these rational numbers are actual zeros.

Example #4 Using the Rational Zero Theorem
(3.3 – Example 3)

Do each of the following for the polynomial function defined by

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2.$$

a) List all possible rational zeros.

Possible zeros: $\frac{p}{q}$, where $p | 2$ and $q | 6$

$$\frac{p}{q} = \frac{\text{factors of } 2}{\text{factors of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6}$$

$$\frac{p}{q} \in \left\{ \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3} \right\}$$

b) Find all rational zeros and factor $f(x)$ into linear factors.

	6	7	-12	-3	2
1	6	13	1	-2	0
-2	6	1	-1	0	

We see that $p(1) = 0 \Rightarrow x-1 \,|\, f(x)$

$f(x) = (x-1)(6x^3 + 13x^2 + x - 2)$

$f(x) = (x-1)(x+2)(6x^2 + x - 1)$

$f(x) = (x-1)(x+2)(3x-1)(2x+1)$

Finding the Rational Zeros of a Polynomial

1. List all possible rational zeros using the Rational Zeros Theorem.
2. Use synthetic division to evaluate the polynomial at each of the candidates for rational zeros that you found in Step 1. when the remainder is 0, note the quotient you have obtained.
3. Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is a quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

Exercise #15 For each polynomial function
(3.3 - #37, 40)

i) list all possible rational zeros;
ii) find all rational zeros
iii) factor $f(x)$.

a) $f(x) = x^3 + 6x^2 - x - 30$.

(i) possible rational zeros $\frac{p}{q} = \frac{\text{factors of } 30}{\text{factor of } 1}$

$$\boxed{\frac{p}{q} \in \{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30\}}$$

(ii)

	1	6	-1	-30
2	1	8	15	0

We see that $p(1) \neq 0$
Try $x = 2$ root

$f(x) = (x-2)(x^2 + 8x + 15)$

(iii) $\boxed{f(x) = (x-2)(x+5)(x+3)}$

The rational zeros are (ii) $\begin{cases} x = 2 \\ x = -5 \\ x = -3 \end{cases}$

b) $f(x) = 15x^3 + 61x^2 + 2x - 8$

(i) Possible rational zeros: $\frac{p}{q} = \frac{\text{factors of } 8}{\text{factors of } 15} = \frac{\pm 1, \pm 2, \pm 4, \pm 8}{\pm 1, \pm 3, \pm 5, \pm 15}$

$\frac{p}{q} \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{3}, \pm \frac{1}{5}, \pm \frac{1}{15}, \pm \frac{2}{3}, \pm \frac{2}{5}, \pm \frac{2}{15}, \pm \frac{4}{3}, \pm \frac{4}{5}, \pm \frac{4}{15}, \pm \frac{8}{3}, \pm \frac{8}{5}, \pm \frac{8}{15}\}$

We see that $p(1) \neq 0$
Try $p(-1) \neq 0$

(ii, iii)

	15	61	2	-8
~~-2~~	~~15~~	~~31~~	~~-60~~	
-4	15	1	-2	0
$\frac{1}{3}$	15	6	0	

$\Rightarrow f(x) = (x+4)(15x^2 + x - 2)$

possible rational zeros
$\cancel{\pm 1}, \cancel{\pm 2}, \pm \frac{1}{3}, \pm \frac{1}{5}, \pm \frac{1}{15}$
$\pm \frac{2}{3}, \pm \frac{2}{5}, \pm \frac{2}{15}$

$\boxed{f(x) = (x+4)(x - \frac{1}{3})(15x + 6)}$ (iii)

The rational zeros are $\boxed{\{-4, \frac{1}{3}, -\frac{2}{5}\}}$ (ii)

Exercise #16 a) Find all the zeros of $f(x) = x^4 - 6x^3 + 22x^2 - 30x + 13$.

Possible rational zeros $\frac{p}{q} = \frac{\text{factors of } 13}{\text{factors of } 1} = \frac{\pm 1, \pm 13}{\pm 1}$ $\frac{p}{q} \in \{\pm 1, \pm 13\}$

x	1	−6	22	−30	13
1	1	−5	17	−13	0
1	1	−4	13	0	

$f(x) = (x-1)(x^3 - 5x^2 + 17x - 13)$

possible zeros: $\pm 1, \pm 13$

$f(x) = (x-1)^2(x^2 - 4x + 13)$

$x^2 - 4x + 13 = 0$

$x = \frac{4 \pm \sqrt{16 - 52}}{2}$

$= \frac{4 \pm \sqrt{-36}}{2}$

$= \frac{4 \pm 6i}{2} = 2 \pm 3i$

The zeros are:

$\begin{cases} x = 1 & \text{zero of multiplicity } 2 \\ x = 2+3i & \text{zero of mult. } 1 \\ x = 2-3i & \text{zero of mult. } 1 \end{cases}$

c) Find all the solutions of $x^4 - 5x^3 - 5x^2 + 23x + 10 = 0$.

Possible rational zeros $\frac{p}{q} = \frac{\text{factors of } 10}{\text{factors of } 1} = \frac{\pm 1, \pm 2, \pm 5, \pm 10}{\pm 1}$

$\frac{p}{q} \in \{\cancel{\pm 1}, \pm 2, \pm 5, \pm 10\}$

we see that $p(1) \neq 0$, $p(-1) \neq 0$

Try $x = 2$

Try $x = 5$

	1	−5	−5	23	10
~~2~~	~~1~~	~~−3~~	~~−11~~	~~1~~	
5	1	0	−5	−2	0
−2	1	−2	−1	0	

$f(x) = (x-5)(x^3 - 5x - 2)$

possible zeros: $\cancel{\pm 1}, \pm 2$

$f(x) = (x-5)(x+2)(x^2 - 2x - 1)$

$x^2 - 2x - 1 = 0$

$x = \frac{2 \pm \sqrt{4+4}}{2}$

$= \frac{2 \pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

The solution set is

$\boxed{\{5, -2, 1 \pm \sqrt{2}\}}$

Descartes' Rule of Signs and Upper and Lower Bounds for Roots

In some cases, the following rule – discovered by the French philosopher and mathematician Rene Descartes around 1637 – is helpful in eliminating candidates from lengthy lists of possible rational roots.

To describe this rule, we need the concept of **variation in sign**. If $f(x)$ is a polynomial with real coefficient, written with descending powers of x (and omitting powers with coefficient 0), then a variation in sign is a change from positive to negative or negative to positive in successive terms of the polynomial (adjacent coefficients have opposite signs).

Example #5 How many variations in sign occur in the following polynomial?

$$f(x) = \underbrace{5x^7 - 3x^5}_{1} \underbrace{- x^4 + 2x^2}_{2} \underbrace{+ x - 3}_{3}$$

Three variations in sign.

Descartes' Rule of Signs
(3.3)

> Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term.
>
> a) The number of positive real zeros of $f(x)$ is either equal to the number of variations in sign in $f(x)$ or is less than that by an even whole number.
>
> b) The number of negative real zeros of $f(x)$ is either equal to the number of variations in sign in $f(-x)$ or is less than that by an even whole number.

Exercise #17 Use Descartes' rule of signs to determine the possible number of positive real zeros and
(3.3 - #73, #77) negative real zeros for each function.

a) $f(x) = \underbrace{2x^3 - 4x^2}_{1} \underbrace{\;+ 2x}_{2} + 7$

Possible # of positive zeros

there are 2 variations in sign in $f(x)$ <
2 positive real zeros
or
$2-2=0$ positive real zeros

Possible # of negative zeros

$f(-x) = -2x^3 - 4x^2 \underbrace{- 2x + 7}_{1}$ one variation in sign => 1 negative real zero

b) $f(x) = x^5 \underbrace{+ 3x^4 - x^3}_{1} \underbrace{\;+ 2x}_{2} + 3$

2 variations in sign in $f(x)$ => {2 positive real zeros OR No positive real zero

$f(-x) = \underbrace{-x^5 + 3x^4}_{1} \underbrace{+ x^3 - 2x}_{2} \underbrace{\;+ 3}_{3}$

3 variations in sign in $f(-x)$ => {3 negative real zeros OR 1 negative real zero

The Upper and Lower Bound Theorem for Real Roots
(3.4 – Boundedness Theorem)

Definition We say that the number b **is a lower bound and** B **is an upper bound** for the roots of a polynomial equation if every root x_i satisfies $b \le x_i \le B$.

Let $f(x)$ be a polynomial of degree $n \ge 1$ with real coefficients and with a positive leading coefficient. If we divide $f(x)$ by $x-c$ using synthetic division and if

1) $c>0$ and the row that contains the quotient and remainder has no negative entry, then c is an upper bound for the real roots of $f(x)=0$ ($f(x)$ has no zero greater than c).

2) $c<0$ and the row that contains the quotient and remainder has entries that alternate in sign – with 0 considered positive or negative, as needed- then c is a lower bound for the real roots of $f(x)=0$ ($f(x)$ has no zero less than c).

Exercise #18 Show that all the real roots of the equation $x^4-3x^2+2x-5=0$ lie between -3 and 2.

We want to show that $x=-3$ is the lower bound $\Rightarrow$ divide $f(x)$ by $x+3$
$x=2$ is the upper bound $\Rightarrow$ divide $f(x)$ by $x-2$

	1	0	-3	2	-5
-3	1	-3	6	-16	43
2	1	2	1	4	3

$\leftarrow$ entries alternate in sign $\Rightarrow x=-3=$ lower bound

$\leftarrow$ all entries positive $\Rightarrow x=2=$ upper bound

Since neither -3 nor 2 is a root all the real roots lie between these numbers.

Exercise #19 Find all the solutions of the equation $2x^5+5x^4-8x^3-14x^2+6x+9=0$.

The possible rational zeros are: $\frac{p}{q}=\frac{\pm1,\pm3,\pm9}{\pm1,\pm2}$ $\frac{p}{q}\in\{\pm1,\pm3,\pm9,\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{9}{2}\}$

We check the positive candidates 1st.
There are 2 variations in sign in $f(x)$, therefore we either have 2 positive roots or none.

	2	5	-8	-14	6	9
1	2	7	-1	-15	-9	0
~~3~~	~~2~~	~~13~~	~~38~~	~~99~~	~~288~~	
$\frac{3}{2}$	2	10	14	6	0	

$x=1$ = root
$P(x)=(x-1)(2x^4+7x^3-x^2-15x-9)$
same possible rational roots.

$\leftarrow$ all entries are positive $\Rightarrow x=3=$ upper bound.

$x=\frac{3}{2}$ = root
$P(x)=(x-1)(x-\frac{3}{2})(2x^3+10x^2+14x+6)$
possible rat. roots, negative
$\{-1,-3,-\frac{1}{2},-\frac{3}{2}\}$

No more positive roots.
We now try the negative roots for $2x^3+10x^2+14x+6$

$$f(-x) = \underbrace{-2x^5 + 5x^4}_{1} + \underbrace{8x^3 - 14x^2}_{2} - \underbrace{6x + 9}_{3}$$

There are either 3 negative roots or 1 negative root.

	2	10	14	6
−1	2	8	6	0
−3	2	2	0	

$x = -1 = \text{root}$

$$f(x) = (x-1)(x-\tfrac{3}{2})(x+1)(2x^2+8x+6)$$

$x = -1$ - not the lower bound

$x = -3 = \text{root}$

$$f(x) = (x-1)(x-\tfrac{3}{2})(x+1)(x+3)(2x+2)$$

$$f(x) = (x-1)(x-\tfrac{3}{2})(x+1)(x+3)\,2(x+1)$$

$$\boxed{f(x) = (x-1)(2x-3)(x+1)^2(x+3)}$$

The zeros are
$$\left\{\begin{array}{ll} x = 1 & \\ x = \frac{3}{2} & \text{of multiplicity } 1 \\ x = -3 & \\ x = -1 & \text{of mult. } 2 \end{array}\right.$$