Activity Lab #2 - Solutions

① $f(x) = -6x^4 - x^3 + 39x^2 - 21x - 20$

ⓐ Since the degree is even and the leading coefficient is negative,
when $x \to -\infty$, $f(x) \to -\infty$ (left)
$x \to \infty$, $f(x) \to -\infty$ (right)

ⓑ

	-6	-1	39	-21	-20
$\frac{4}{3}$	-6	-9	27	15	0

so $f(x) = \left(x - \frac{4}{3}\right)(-6x^3 - 9x^2 + 27x + 15)$

$f(x) = \left(x - \frac{4}{3}\right)(-3)(2x^3 + 3x^2 - 9x - 5)$

$f(x) = -(3x-4)(2x^3 + 3x^2 - 9x - 5)$

ⓒ $f(0) = -20$
$f(-1) = 35$

Therefore, since f changes sign on the interval $[-1, 0]$ there is some r between -1 and 0 such that $f(r) = 0$

ⓓ Since the coefficients of f are all integers, we may conclude that the rational zeros of f are in the set $\pm\left\{1, 2, 4, 5, 10, 20, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{2}{3}, \frac{4}{3}, \frac{5}{2}, \frac{5}{3}, \frac{5}{6}, \frac{10}{3}, \frac{20}{3}\right\}$

ⓔ

	-6	-1	39	-21	-20
$-\frac{2}{3}$	-6	3	37	$-\frac{137}{3}$	$\frac{94}{9}$

$f(x) = \left(x + \frac{2}{3}\right)\left(-6x^3 + 3x^2 + 37x - \frac{137}{3}\right) + \frac{94}{9}$

(f)

	-6	-1	39	-21	-20
$\frac{5}{2}$	-6	-16	-1	$-\frac{47}{2}$	$-\frac{315}{4}$

$$f(x)=\left(x-\tfrac{5}{2}\right)\left(-6x^3-16x^2-x-\tfrac{47}{2}\right)-\tfrac{315}{4}$$

so, if $x>\frac{5}{2}$, $f(x)<-\frac{315}{4}$ so can't be zero

(g)

$$\begin{array}{r l}
 & -6x^2+5x+4 \\
x^2+x-5 & \overline{\big)\,-6x^4-x^3+39x^2-21x-20} \\
 & +6x^4+6x^3-30x^2 \\
\hline
 & \quad 5x^3+9x^2-21x-20 \\
 & \quad -5x^3-5x^2+25x \\
\hline
 & \qquad 4x^2+4x-20 \\
 & \qquad -4x^2-4x+20 \\
\hline
 & \qquad\qquad 0
\end{array}$$

$$f(x)=(x^2+x-5)(-6x^2+5x+4)$$

(h) From (6) $\Rightarrow$ $f(x)=-(3x-4)(2x^3+3x^2-9x-5)$

Possible rational zeros of $2x^3+3x^2-9x-5$ are in the set $\pm\{1, 5, \frac{1}{2}, \frac{5}{2}\}$

	2	3	-9	-5
$-\frac{1}{2}$	2	2	-10	0

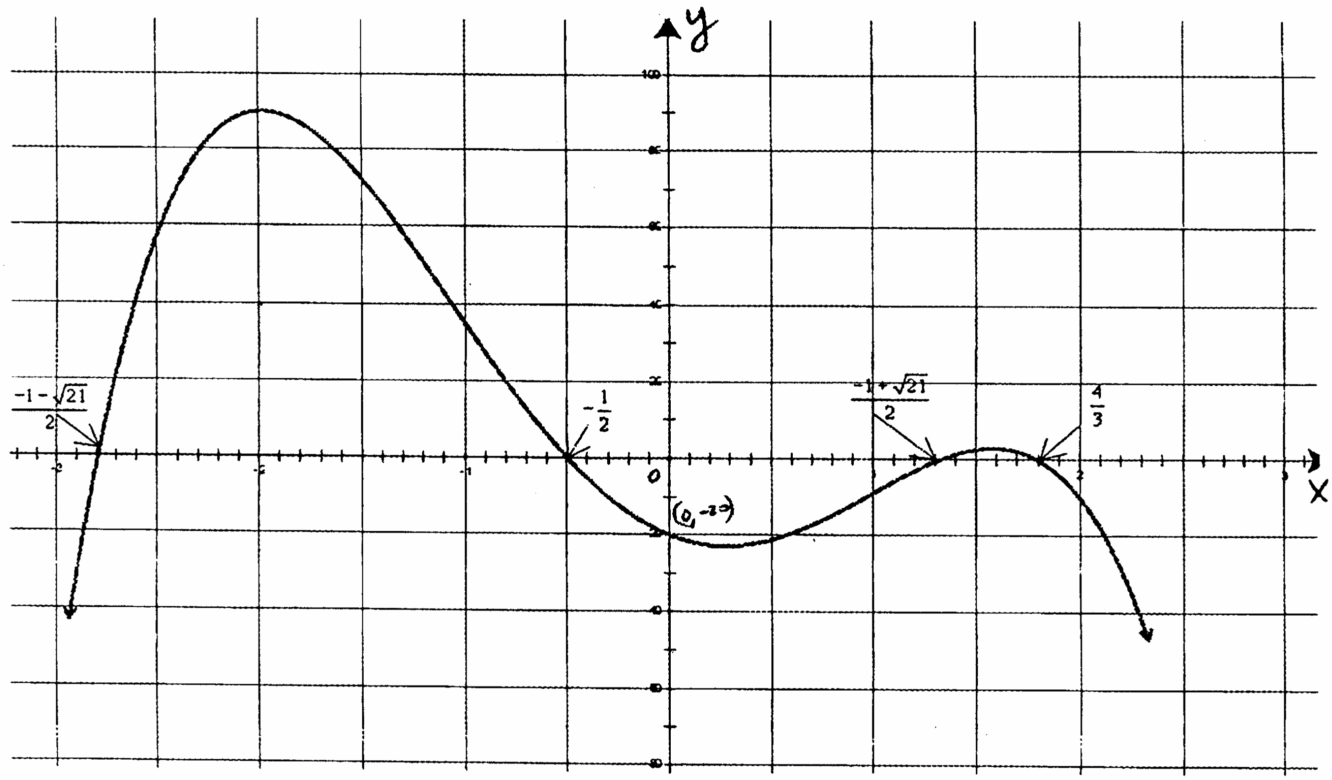
so, $f(x)=-(3x-4)\left(x+\frac{1}{2}\right)(2x^2+2x-10)$

$f(x)=-(3x-4)(2x+1)(x^2+x-5)$

$$x^2+x-5=0$$

$$x_{1,2}=\frac{-1\pm\sqrt{21}}{2}$$

$$f(x)=-(3x-4)(2x+1)\left(x-\frac{-1+\sqrt{21}}{2}\right)\left(x-\frac{-1-\sqrt{21}}{2}\right)$$



(2) $x=-4$ root of multiplicity 3
$x=-\frac{3}{2}$ root of multiplicity 2
$x=1$ root of multiplicity 2

So, $f(x) = a(x+4)^3\left(x+\frac{3}{2}\right)^2(x-1)^2$

and $f(0) = 12$, so

$$12 = a(4)^3\left(\frac{3}{2}\right)^2(-1)^2$$

$$12 = 144a$$

$$a = \frac{12}{144}$$

$$a = \frac{1}{12}$$

So, $f(x) = \frac{1}{12}(x+4)^3\left(x+\frac{3}{2}\right)^2(x-1)^2$

OR

$f(x) = \frac{1}{48}(x+4)^3(2x+3)^2(x-1)^2$

(3) $x-1 \mid x^3+x^2+ax+b$

$$\begin{array}{c|cccc} & 1 & 1 & a & b \\ \hline 1 & 1 & 2 & 2+a & 2+a+b \end{array}$$

iff $2+a+b=0$

and $x-1 \mid x^3-x^2-ax+b$

$$\begin{array}{c|cccc} & 1 & -1 & -a & b \\ \hline 1 & 1 & 0 & -a & -a+b \end{array}$$

iff $1-a+b=0$

$$\begin{cases} a+b=-2 \\ -a+b=0 \end{cases}$$

(+) $2b=-2 \Rightarrow b=-1 \Rightarrow a=-1$

(4) $x^3+x^2+x-p=0$; let $f(x)=x^3+x^2+x-p$

Possible rational zeros: $\pm 1, \pm p$

$x=1$ zero $\Leftrightarrow f(1)=0 \Leftrightarrow 1+1+1-p=0 \Leftrightarrow \boxed{p=3}$

$x=-1$ zero $\Leftrightarrow f(-1)=0 \Leftrightarrow -1+1-1-p=0 \Leftrightarrow p=-1$ not prime

$x=p$ zero $\Leftrightarrow f(p)=0 \Leftrightarrow p^3+p^2+p-p=0$

$p^2(p+1)=0 \Rightarrow p=0,\ p=-1$ not prime

$x=-p$ zero $\Leftrightarrow f(-p)=0 \Leftrightarrow -p^3+p^2-p-p=0$

$p^3-p^2+2p=0$

$p(p^2-p+2)=0$

$p=0$ not prime

Therefore, $\boxed{p=3}$

$x^3+x^2+x-3=0$

$$\begin{array}{c|cccc} & 1 & 1 & 1 & -3 \\ \hline 1 & 1 & 2 & 3 & 0 \end{array}$$

$(x-1)(x^2+2x+3)=0$

$x^2+2x+3=0$

$x=-1\pm\sqrt{2}$

The roots of the equation are $\{1, -1\pm\sqrt{2}\}$

(5) $x^3 - 3x^2 + 3x - 26 = 0$ Let $f(x) = x^3 - 3x^2 + 3x - 26$

x	$f(x)$
3	-17
4	2
3.9	-0.6

$f(3) < 0$ and $f(4) > 0$, therefore, since $f(x)$ changes sign on the interval $[3,4]$ there is some $r \in (3,4)$ such that $f(r) = 0$

Moreover, $f(3.9) < 0$ and $f(4) > 0$, therefore, since $f(x)$ changes sign on $[3.9, 4.0]$, there is a root $\boxed{r \in (3.9, 4)}$ such that $f(r) = 0$.

(6) $$f(x) = \frac{x^3 + 6x^2 + 9x}{2x^3 - 2}$$

a) $$f(x) = \frac{x(x^2 + 6x + 9)}{2(x^3 - 1)} = \frac{x(x+3)^2}{2(x-1)(x^2+x+1)}$$

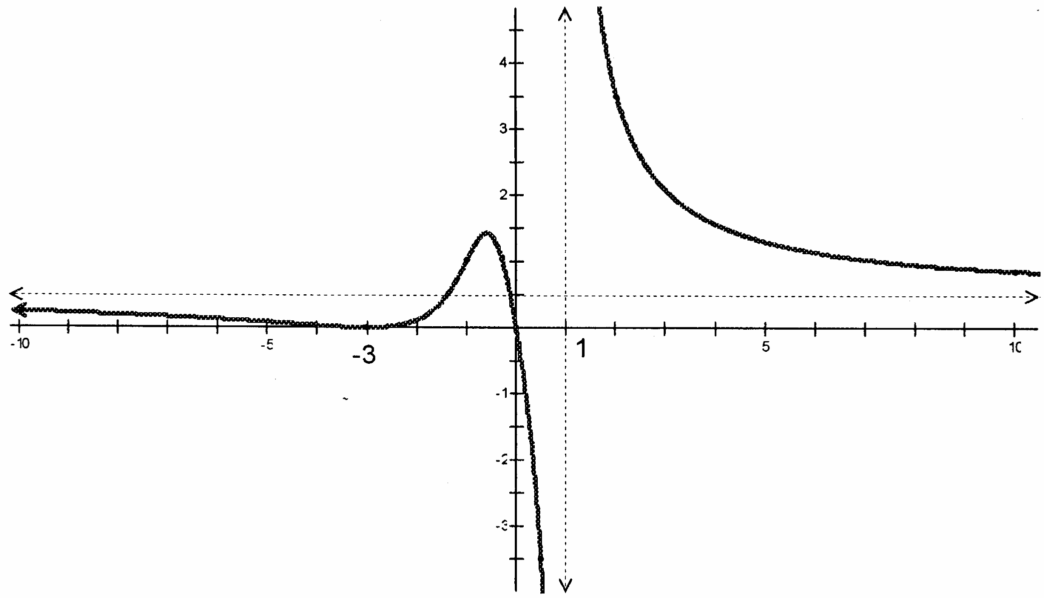
b) x-intercepts: $f(x) = 0 \iff x(x+3)^2 = 0$ $\boxed{\begin{matrix}(0,0) \\ (-3,0)\end{matrix}}$ y-n, x-n, x-n.

c) The vertical asymptote is along $\boxed{x = 1}$

d) The horizontal asymptote is along $\boxed{y = \frac{1}{2}}$

e)

x	$-\infty$	-10	-3	-1	0	0.5	1	2	10	∞
$f(x)$	$\frac{1}{2}$	0.24	0	1	0	-3.5	$-\infty \mid \infty$	3.6	0.85	$\frac{1}{2}$
	$\boxed{y = \frac{1}{2}}$ H.A						$\boxed{x = 1}$ V.A			$\boxed{y = \frac{1}{2}}$ H.A



(7) $$y = \frac{(x^2-4)(2x^2-1)}{(x-1)^2(x^2+x+1)} = \frac{(x-2)(x+2)(2x^2-1)}{(x-1)^2(x^2+x+1)}$$

x	$-\infty$	-2	-1	-0.7	0	0.7	0.8	1	1.5	2	6	∞
y	2	0	$-\frac{3}{4}$	0	4	0	-9.6	$-\infty$ | $-\infty$	-5.1	0	2.11	2
	$\boxed{y=2}$ H.A							$\boxed{x=1}$ V.A				$\boxed{y=2}$ H.A

$x \in \mathbb{R} \setminus \{1\}$ $\quad \boxed{x=1}$ V.A.

Horizontal asymptote: $\boxed{y=2}$ H.A.

x-intercepts:
$x = 2$
$x = -2$
$2x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{\sqrt{2}}{2} \simeq \pm 0.7$

y-intercept: $x=0$, $y = \frac{(-4)(-1)}{(1)(1)} = 4$

Behavior near the vertical asymptote $x = 1$

$x \to 1^+$, $y \to \frac{(-1)(3)(1)}{(+0)(1)}$, $y \to -\infty$; $x \to 1^-$, $y \to \frac{(-1)(3)(1)}{(+0)(1)}$, $y \to -\infty$

OR take test points: $x = 1.5$, $y = -5.1$; $x = 0.8$, $y = -9.6$

Other test points

$x = -1$, $y = \frac{(-3)(1)}{4(1)} = \frac{-3}{4}$