Exercise #1:

(a) Graph the following parabola: $y = x^2 + 3x + 2$

$a > 0 \Rightarrow$ parabola opening upward

Vertex:

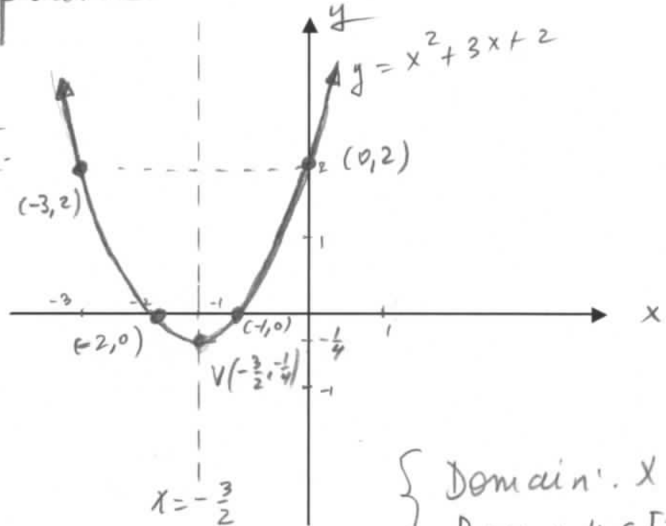
$V(x_v, y_v)$

$x_v = \frac{-b}{2a} = \frac{-3}{2(1)} = \frac{-3}{2}$

then $y_v = \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) + 2$

$y_v = -\frac{1}{4}$

$\boxed{V\left(-\frac{3}{2}, -\frac{1}{4}\right)}$



$\left\{ \begin{array}{l} \text{Domain: } x \in \mathbb{R} \\ \text{Range: } y \in \left[-\frac{1}{4}, \infty\right) \end{array} \right.$

y-intercept:

let $x = 0$
then $y = 2$

$\boxed{0, 2}$

x-intercepts:

let $y = 0$
then $x^2 + 3x + 2 = 0$
$(x+1)(x+2) = 0$
$x = -1, \ x = -2$

$\boxed{(-1,0) \text{ and } (-2,0)}$

(b) Graph the following parabola: $y = -2x^2 + 4x + 1$

$a < 0 \Rightarrow$ parabola opening downward

$\left\{ \begin{array}{l} \text{Domain: } x \in \mathbb{R} \\ \text{Range: } y \le 3 \end{array} \right.$

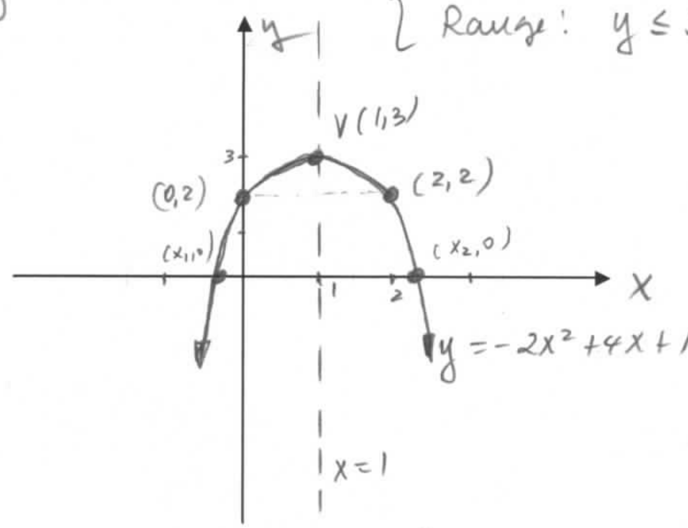
Vertex:

$V(x_v, y_v)$

$x_v = \frac{-b}{2a} = \frac{-4}{2(-2)} = 1$

then $y_v = -2 + 4 + 1 = 3$

$\boxed{V(1,3)}$



y-intercept:

let $x = 0$
then $y = 2$

$\boxed{(0,2)}$

x-intercepts:

let $y = 0$
then $-2x^2 + 4x + 1 = 0$
$2x^2 - 4x - 1 = 0$

$x = \frac{4 \pm \sqrt{16 - 4(2)(-1)}}{2(2)} = \frac{4 \pm \sqrt{24}}{4} = \frac{4 \pm 2\sqrt{6}}{4}$

$= \frac{2(2 \pm \sqrt{6})}{4}$

$= \frac{2 \pm \sqrt{6}}{2}$

$\boxed{\left(\frac{2 \pm \sqrt{6}}{2}, 0\right)}$

$\approx \frac{2 \pm 2.5}{2} \left\{ \begin{array}{l} 2.25 \ (x_2) \\ -0.25 \ (x_1) \end{array} \right.$

(c) Graph the following parabola: $y = x^2 + x - 6$ - parabola opening up $(a=1>0)$

Vertex:

$x_v = \frac{-b}{2a} = \frac{-1}{2}$

$\boxed{V\left(\frac{-1}{2}, \frac{-25}{4}\right)}$ $y_v = \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right) - 6$

$= \frac{1}{4} - \frac{1}{2} - \frac{6}{1} =$

$= \frac{1-2-24}{4} = \frac{-25}{4}$

y-intercept:

let $x=0$, $y=-6$

$\boxed{(0,-6)}$

x-intercepts:

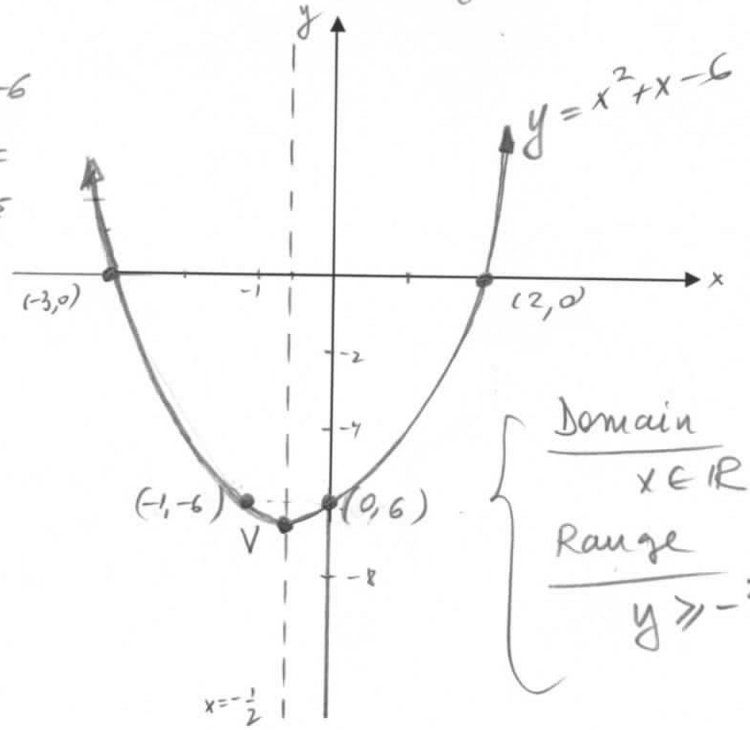
let $y=0$,

$x^2+x-6=0$

$(x+3)(x-2)=0$

$x=-3$, $x=2$

$\boxed{(-3,0) \text{ and } (2,0)}$



Domain
$x \in \mathbb{R}$
Range
$y \geqslant -\frac{25}{4}$

The Vertex Form of a Parabola

Example $y = -2x^2 + 4x + 1$

Complete the square on x:

1st $y = -2(x^2 - 2x) + 1$

2nd $\left(\frac{1}{2} \text{ coef.} x\right)^2 = \left(\frac{1}{2}(-2)\right)^2 = 1$

3rd $y = -2(x^2 - 2x + 1) + 1 + \boxed{2}$

$\boxed{-2}$

$\boxed{y = -2(x-1)^2 + 3}$

vertex is $V(1,3)$

The Vertex Form of a Parabola: $\boxed{y = a(x - x_v)^2 + y_v}$, where $V(x_v, y_v)$ is the vertex and a is the coefficient of x^2.

Exercise #2: Find the vertex of each parabola. Decide whether the vertex is a maximum or a minimum point.

$y = a(x - x_v)^2 + y_v$

(a) $y = 2(x-3)^2 + 4$

$V(3,4)$ minimum, as the parabola opens upward $(a = 2 > 0)$

(b) $y = -3(x+3)^2 - 5$

$V(-3,-5)$ maximum; it's a parabola that opens downward $(a = -3 < 0)$

(c) $y = 3x^2 + 4x + 2$

Method I – using the vertex formula

$x_v = -\frac{b}{2a}$

$= \frac{-4}{2(3)} = \frac{-2}{3}$

$y_v = 3\left(-\frac{2}{3}\right)^2 + 4\left(-\frac{2}{3}\right) + 2$

$= 3 \cdot \frac{4}{9} - \frac{8}{3} + 2$

$= \frac{4}{3} - \frac{8}{3} + \frac{2}{1} = \frac{-4+6}{3} = \frac{2}{3}$

$\boxed{V\left(-\frac{2}{3}, \frac{2}{3}\right)}$

Method II – completing the square on x

$y = 3x^2 + 4x + 2$

1st $y = 3\left(x^2 + \frac{4}{3}x\right) + 2$

2nd $\left(\frac{1}{2} \text{ coef } x\right)^2 = \left(\frac{1}{2} \cdot \frac{4}{3}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

3rd $y = 3\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 2 \boxed{-\frac{4}{3}}$

$3 \cdot \frac{4}{9} = \boxed{\frac{4}{3}}$

$\boxed{y = 3\left(x + \frac{2}{3}\right)^2 + \frac{2}{3}}$

so $\boxed{V\left(-\frac{2}{3}, \frac{2}{3}\right)}$

Exercise #3: Write an equation for a parabola with vertex $V(1,-3)$.

$\left.\begin{array}{l} y = a(x - x_v)^2 + y_v \\ x_v = 1,\ y_v = -3 \\ \text{let } a = 1 \end{array}\right\} \Rightarrow y = 1 \cdot (x-1)^2 - 3$

$\boxed{y = (x-1)^2 - 3}$

Exercise #4: Write an equation for the parabola with vertex $V(-1,1)$ that passes through the point $(2,3)$.

$\left.\begin{array}{l} y = a(x - x_v)^2 + y_v \\ x_v = -1,\ y_v = 1 \end{array}\right\} \Rightarrow y = a(x+1)^2 + 1$

$(2,3) \in$ graph $\Rightarrow$ when $x = 2$, $y = 3$

$3 = a(2+1)^2 + 1$

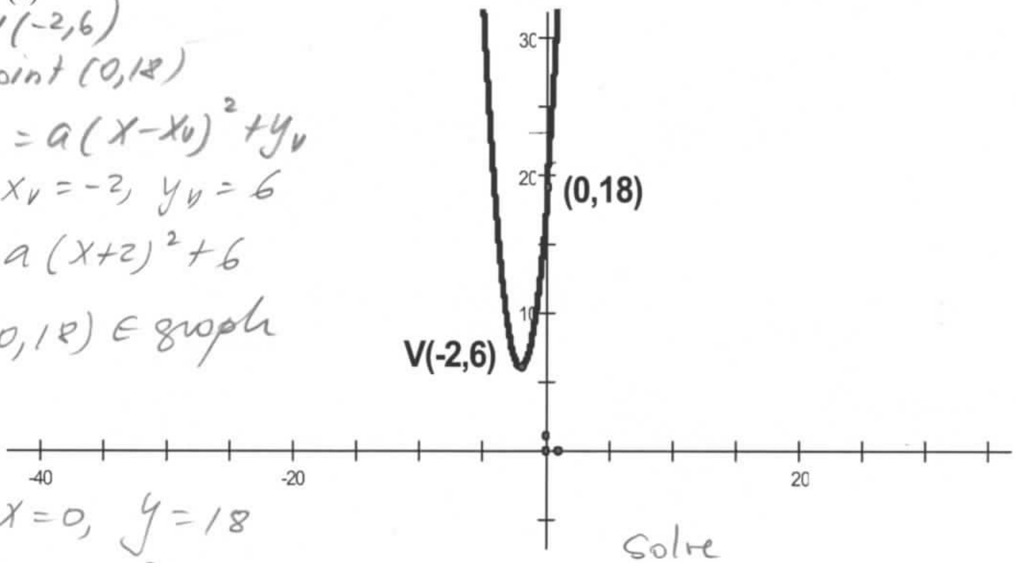
$3 = 9a + 1 \Rightarrow a = \frac{2}{9}$

$\boxed{y = \frac{2}{9}(x+1)^2 + 1}$

Exercise #5: i) Write an equation for each graph.
ii) What is the domain and the range of the function?
iii) Using the graph, solve the following: $f(x)>0$; $f(x)=0$; $f(x)<0$.

(i)

(a)



given: $V(-2,6)$
point $(0,18)$

know: $y = a(x-x_v)^2 + y_v$

$x_v = -2$, $y_v = 6$

$y = a(x+2)^2 + 6$

given: $(0,18) \in$ graph

so

when $x=0$, $y=18$

$18 = a(2)^2 + 6$

$18 = 4a + 6$

$12 = 4a \Rightarrow a = 3$

so $\boxed{y = 3(x+2)^2 + 6}$

(ii) $\boxed{\begin{array}{l} x \in \mathbb{R} \\ y \geq 6 \end{array}}$

(iii) Solve $f(x) > 0$ means that we need to find x such that $y > 0$

$\boxed{\text{Any } x \in \mathbb{R}}$ satisfies the condition

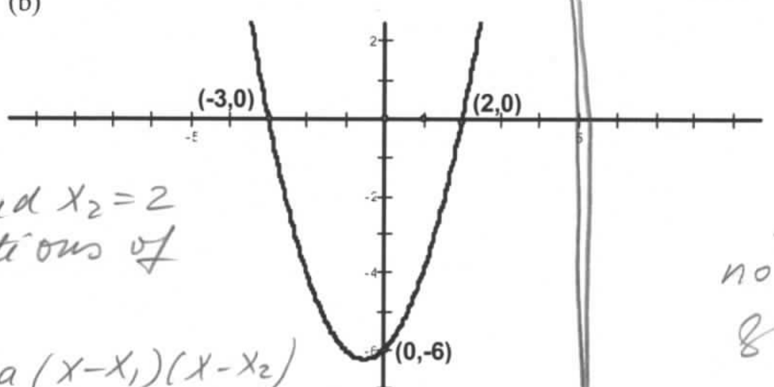
Solve $f(x) = 0$

find x such that $y = 0$

no x-intercepts $(y=0)$

so $\boxed{x \in \emptyset}$

(b)



(i) given:

x-n: $(-3,0)$
$(2,0)$

$\Rightarrow x_1 = -3$ and $x_2 = 2$ are solutions of $y = 0$

Know: $y = a(x-x_1)(x-x_2)$

$y = a(x+3)(x-2)$

Solve $f(x) < 0$

find x such that $y < 0$

no points on the graph below the x-axis

so $\boxed{x \in \emptyset}$

given: $(0,-6) \in$ graph

$x = 0,\ y = -6$

$-6 = a(3)(-2)$

$a = 1$

$\therefore\ y = (x+3)(x-2)$

$\boxed{y = x^2 + x - 6}$

(ii) $\boxed{x \in \mathbb{R}}$

$y \geq y_v$

find y_v

$x_v = \frac{-b}{2a} = \frac{-1}{2}$

$y_v = \frac{1}{4} - \frac{1}{2} - 6$

$y_v = \frac{-25}{4}$, $\boxed{y \geq \frac{-25}{4}}$

(iii) $f(x) > 0$ iff

$x \in (-\infty, -3) \cup (2, \infty)$

$f(x) = 0$ iff

$x \in \{-3, 2\}$

$f(x) < 0$ iff

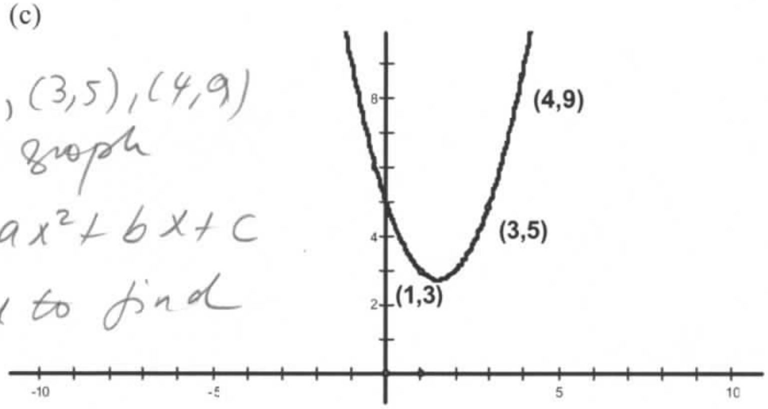
$x \in (-3, 2)$

(c)

given $(1,3), (3,5), (4,9)$
$\in$ graph

find $y = ax^2 + bx + c$

$\therefore$ we need to find
a, b, c



$(1,3) \in$ graph $\Rightarrow x = 1, y = 3$

$a + b + c = 3$ (1)

$(3,5) \in$ graph $\Rightarrow x = 3, y = 5$

$9a + 3b + c = 5$ (2)

$(4,9) \in$ graph $\Rightarrow x = 4, y = 9$

$16a + 4b + c = 9$ (3)

Solve the system

$$\begin{cases} a + b + c = 3 & (1) \\ 9a + 3b + c = 5 & (2) \\ 16a + 4b + c = 9 & (3) \end{cases}$$

Elimination method
we'll elim. c

$$\begin{aligned} (2)\ & 9a + 3b + c = 5 \\ (1)\ & a + b + c = 3 \\ \hline - \ & 8a + 2b = 2 \quad / \div 2 \\ & 4a + b = 1 \quad (4) \end{aligned}$$

$$\begin{aligned} (3)\ & 16a + 4b + c = 9 \\ (2)\ & 9a + 3b + c = 5 \\ \hline - \ & 7a + b = 4 \quad (5) \end{aligned}$$

Solve $\begin{cases} 4a + b = 1 & (4) \\ 7a + b = 4 & (5) \end{cases}$

$-\quad -3a = -3 \Rightarrow a = 1$

Substitute $a = 1$ into (4) $\Rightarrow$

$4 + b = 1 \Rightarrow b = -3$

Substitute $a = 1, b = -3$ into (1)

$1 - 3 + c = 3$

$-2 + c = 3 \Rightarrow c = 5$

The equation is

$\boxed{y = x^2 - 3x + 5}$

$s(t) = -16t^2 + 64t + 160$

$\begin{cases} t = \text{time (s)} \\ s(t) = \text{height of ball above ground (ft)} \end{cases}$

Applications:

(1) A person standing close to the edge on the top of a 160-foot building throws baseball vertically upward. The quadratic function $s(t) = -16t^2 + 64t + 160$ models the ball's height above the ground, *s(t)*, in feet, *t* seconds after it is thrown.

(a) After how many seconds does the ball reach its maximum height? What is the maximum height?

The equation represents a parabola opening down, therefore the maximum occurs at the vertex $V(t_V, S_V)$

$t_V = \frac{-b}{2a} = \frac{-64}{2(-16)} = 2$ seconds; $S_V = -16(4) + 64(2) + 160 = 224$ feet

It takes 2 seconds to reach the max. height of 224 ft.

(b) How many seconds does it take until the ball finally hits the ground?

find t when $s(t) = 0$

$-16t^2 + 64t + 160 = 0 \quad / \div(-16)$

$t^2 - 4t - 10 = 0$

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$t = \frac{4 \pm \sqrt{56}}{2}$ $\begin{cases} 5.74 \text{ seconds} \\ - \text{ not positive} \end{cases}$

It takes the ball about 5.74 seconds to reach the ground

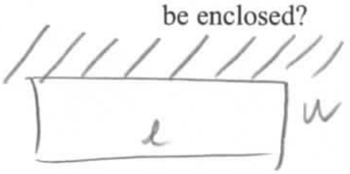
(c) Find *s*(0) and describe what it means.

$s(0) = 160$ feet

it represents the initial height of the ball

initially, the ball was at 160 ft above the ground

(2) You have 600 feet of fencing to enclose a rectangular plot that borders on a river. If you do not fence the side along the river, find the length and width of the plot that will maximize the area. What is the largest area that can be enclosed?



let l = length

w = width

given: $2w + l = 600$ ft (1)

want the area $= wl$ to be maximum

let A = area $\Rightarrow A = wl$ (2)

(1) $2w + l = 600 \Rightarrow$

$l = 600 - 2w$ (3)

substitute (3) into (2):

$A = w(600 - 2w)$

$A = -2w^2 + 600w$

$A = -2w^2 + 600w$

represents a parabola opening down, so the max. occurs at the vertex $V(W_V, A_V)$

$W_V = \frac{-b}{2a} = \frac{-600}{2(-2)} = 150$ ft (4)

substitute (4) into (3):

$l = 600 - 2(150)$

$l = 300$ ft.

The dimensions should be

$\boxed{\begin{array}{l} w = 150 \text{ ft} \\ l = 300 \text{ ft} \end{array}}$ in order to maximize the area

$A_{max} = 150\text{ft} \cdot 300\text{ ft}$

$A_{max} = 45{,}000 \text{ ft}^2$

(3) Virtual Fido is a company that makes electronic virtual pets. The fixed weekly cost is \$3000, and variable costs for each pet is \$20.

(a) Let x represent the number of virtual pets made and sold each week. Write the weekly cost function, C, for Virtual Fido.

$$\boxed{C(x) = 3000 + 20x}$$

(b) The function $R(x) = -x^2 + 1000x$ describes the money that Virtual Fido takes in each week from the sale of x virtual pets. Use this revenue function and cost function from part (a) to write the weekly profit function P.

$P(x) = R(x) - C(x)$

$P(x) = -x^2 + 1000x - (3000 + 20x) \Rightarrow \boxed{P(x) = -x^2 + 980x - 3000}$

(c) Use the profit function to determine the number of virtual pets that should be made and sold each week to maximize the profit. What is the maximum weekly profit?

$P(x) = -x^2 + 980x - 3000$

parabola opening downward $(a = -1 < 0)$

so its maximum occurs at the vertex $V(x_V, P_V)$

$x_V = \frac{-b}{2a} = \frac{-980}{2(-1)} = 490$ virtual pets

$P_V = P_{max} = -(490)^2 + 980(490) - 3000$

$= 237{,}100$ \$

They should make and sell 490 pets in order to obtain a max. profit of 237,100 \$

More applications

(4) The total profit Kiyoshi makes from producing and selling "x" floral arrangements is

$$P = -0.4x^2 + 36x.$$

a) How many floral arrangements should Kiyoshi produce and sell to maximize his profit?
b) What is his maximum profit? Explain how you know for sure you found the maximum profit.

(5) The sum of twice one number and three times another number is 5. Find two such numbers so to maximize their product.

(6) During a single day, the Roll-It shop will rent 35 rollerblades if they charge \$5 per rental. They find that for every 20 cents they increase the charge, they lose one rental.

a) Find an expression for the total revenue "R" in terms of the number of \$0.20 price increases "x".
b) What is the amount they can charge in order to maximize their revenue?

(7) The owners of a small fruit orchard decide to produce gift baskets as a sideline. The cost per basket for producing x baskets is $C = 0.01x^2 - 2x + 120$. How many baskets should they produce in order to minimize the cost per basket? What will their total cost be at that production level?

MORE APPLICATIONS

(4) $P = -0.4x^2 + 36x$

$\begin{cases} x = \text{number of floral arrangements} \\ P = \text{total profit when making and selling } x \text{ units.} \end{cases}$

The equation represents a parabola opening down, so the max. occurs at the vertex $V(x_v, P_v)$

$$x_v = \frac{-b}{2a} = \frac{-36}{2(-0.4)} = \boxed{45} \text{ floral arrangements}$$

$$P_v = -0.4(45)^2 + 36(45) = \boxed{810\ \$} \text{ profit}$$

They should make and sell 45 units to obtain a maximum profit of 810 \$

(5) let x = one number
y = the other number

then $2x + 3y = 5$ (1)

Want their product, say P, to be maximum

$P = xy$ (2)

(1) $2x + 3y = 5 \Rightarrow 2x = 5 - 3y$

$$x = \frac{5}{2} - \frac{3}{2}y \quad (3)$$

Substitute (3) into (2) $\Rightarrow$

$$P = \left(\frac{5}{2} - \frac{3}{2}y\right)y$$

$\boxed{P = -\frac{3}{2}y^2 + \frac{5}{2}y}$. parabola opening down; max occurs at the vertex $V(y_v, P_v)$

$$y_v = \frac{-b}{2a} = \frac{-\frac{5}{2}}{2(-\frac{3}{2})} = \frac{\frac{5}{2}}{3} = \frac{5}{6} \quad (4)$$

Substitute (4) into (3) $\Rightarrow$

$$x = \frac{5}{2} - \frac{3}{2} \cdot \frac{5}{6}$$

$$= \frac{5}{2} - \frac{5}{4} = \frac{5}{4}$$

So the numbers should be $x = \frac{5}{4}$ and $y = \frac{5}{6}$

(6) 35 units if $5/unit

let x = number of 0.20 $ price increments

(a) Revenue = (price per unit) · (total # of units)

price per unit = $5 + 0.20x$

total # of units = $35 - x$

so $R = (5+0.20x)(35-x)$ - revenue

$\boxed{R = -0.2x^2 + 2x + 175}$ - revenue

(b) The revenue equation represents a parabola opening down with max. at the vertex $V(x_v, R_v)$

$x_v = \frac{-b}{2a} = \frac{-2}{2(-0.2)} = 5$ increments of twenty cents

so they should charge $5 + 0.2x = 5 + 0.2(5) = 6$ $/unit

(7) $C = 0.01x^2 - 2x + 120$

x = number of baskets

C = cost per basket (when producing x baskets)

The eq. represents a parabola opening up, therefore the minimum occurs at the vertex $V(x_v, C_v)$

$x_v = \frac{-b}{2a} = \frac{-(-2)}{2(0.01)} = \boxed{100 \text{ baskets}}$

$C_v = 0.01(100)^2 - 2(100) + 120 = \boxed{20 \text{ \$/basket}}$

They should produce 100 baskets in order to minimize the cost per basket. The min. cost per basket is 20 $

Total cost $= 20 \frac{\$}{\text{basket}} \cdot 100 \text{ baskets} = \boxed{2000 \text{ \$}}$