2.1 Introduction to Functions
2.2 The Algebra of Functions

The word *function*, used casually, expresses the notion of dependence. For example, a person might say that election results are a function of the economy, meaning that the winner of an election is determined by how the economy is doing. Another person may claim that car sales are a function of the weather, meaning that the number of cars sold on a given day is affected by the weather.

In mathematics, the meaning of the word *function* is more precise, but the basic idea is the same.

Definition 1

A function is a relationship between two quantities. If the value of one quantity uniquely determines the value of the second quantity, we say the second quantity is a function of the first.

Example 1 In the early 1980s, the recording industry introduced the compact disc (CD) as an alternative to vinyl long playing records (LPs). Table 1 gives the number of units (in millions) of CDs sold for the years 1982 through 1987. The year uniquely determines the number of CDs sold. Thus, we say that the number of CDs sold is a function of the year. We may also say that the number of CDs sold depends on the year.

TABLE 1
Millions of CDs sold, by year

Year	Sales (millions)
1982	0
1983	0.8
1984	5.8
1985	23
1986	53
1987	102

Note: The quantities described by a function are called *variables.*

Definition 2

A function is a relationship between two variables: **independent variable (input)** and **dependent variable (output)** that assigns to each independent variable a unique value of the dependent variable.

Function Notation

There is a convenient notation we use when discussing functions. First, we choose a name for the particular function, let's say f. That is, f is the name of the relationship between the two variables. let's say t (the year) and S (the CD's sold) from Example 1. Then we can write

$$S = f(t)$$

which means " S is a function of t, and f is the name of the function."

Domain and Range

If $y = f(x)$, then
- the **domain** of f is the set of values for the independent variable, x
- the **range** of f is the set of values for the dependent variable, y

Functions Defined by Tables

Example 2 a) Is C a function of M? Explain.

TABLE 2
Millions of CDs sold, by year

Cost of merchandise (M)	**Shipping Charges (C)**
\$0.01 – 10.00	\$2.50
10.01 – 20.00	3.75
20.01 – 30.00	4.85
30.01 – 50.00	5.95
50.01 – 75.00	7.95
Over 75.00	8.95

b) Is M a function of C? Explain.

c) If $C = f(M)$, find $f(3)$. What does it mean?

d) Solve $f(M) = 6.95$. What does it mean?

Example 3 Tables 3a, 3b, and 3c represent the relationship between the button number, *N*, which you push, and the snack, *S*, delivered by three different vending machines.

Table 3a
Vending Machine #1

N	*S*
1	m&ms
2	pretzels
3	dried fruit
4	Hersheys
5	fat free cookies
6	Snickers

Table 3b
Vending Machine #2

N	*S*
1	m&ms or dried fruit
2	Pretzels or Hersheys
3	Snickers or fat free cookies

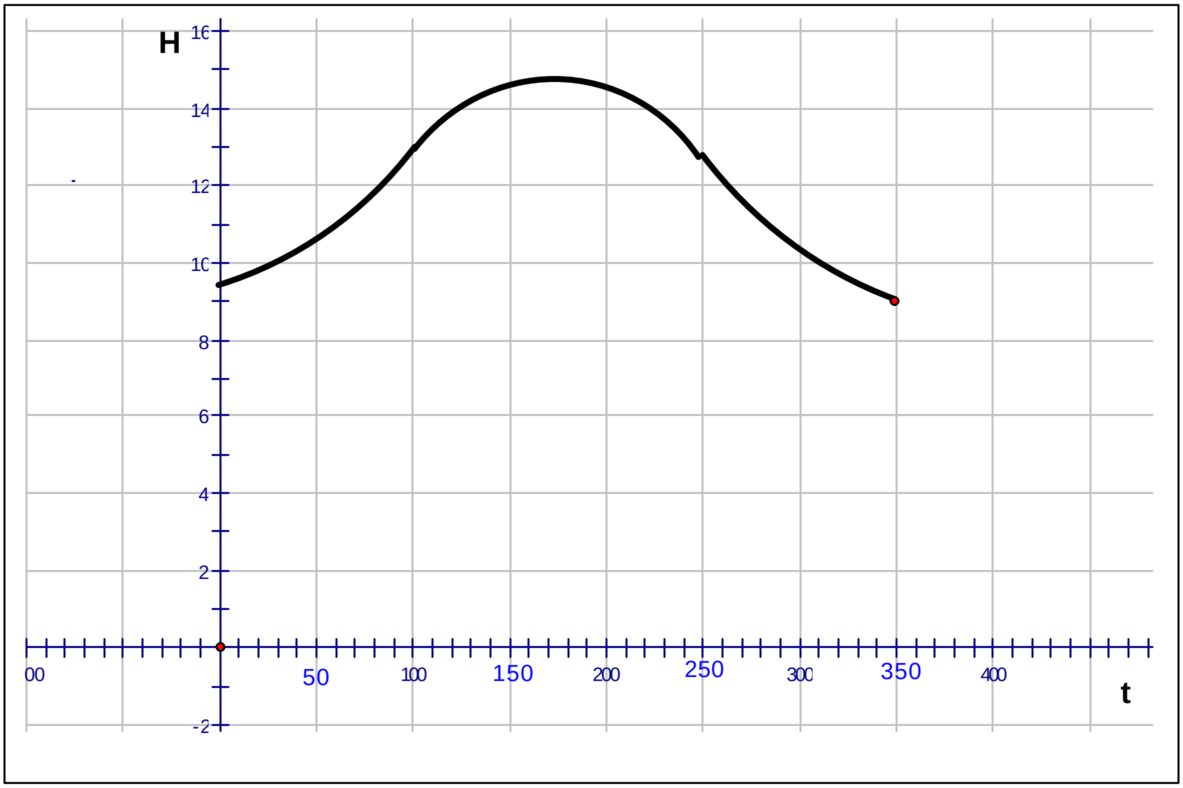
Table 3c
Vending Machine #3

N	*S*
1	m&ms
2	m&ms
3	dried fruit
4	Hersheys
5	Hersheys
6	fat free cookies
7	Snickers
8	Snickers

One of these vending machines is not a good one to use, because *S* is not a function of *N*. Which one? Explain why this makes it a bad machine to use.

Functions Defined By Graphs

Example 4 Figure 4 shows the number of hours, H, that the sun is above the horizon in Peoria, Illinois, on day t, where January 1 corresponds to $t = 0$.



a) Which variable is independent, and which is dependent?

b) Approximately how many hours of sunlight are there in Peoria on day 150?

c) On which days are there 12 hours of sunlight?

d) What are the maximum and minimum values of H, and when do these values occur?

Exercise 1 (textbook 2.1)

In Exercises 37-42, use the graph of f to find each indicated function value.

37. $f(-2)$

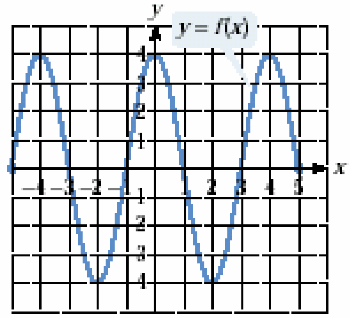
38. $f(2)$

39. $f(4)$

40. $f(-4)$

41. $f(-3)$

42. $f(-1)$



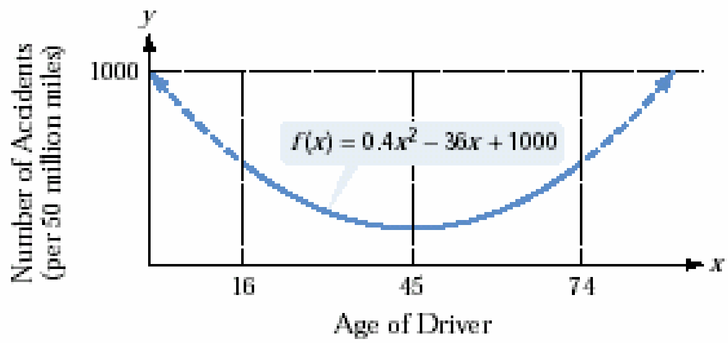
a) Is y a function of x? Explain.

b) Find the domain and range of f.

Exercise 2 (textbook 2.1)

The function $f(x) = 0.4x^2 - 36x + 1000$

Models the number of accidents, $f(x)$, per 50 million miles driven as a function of a driver's age, x, in years, where x includes drivers from ages 16 through 74, inclusive. Answer the following:



a) Which variable is independent and which one is dependent?

b) Find and interpret $f(20)$. Identify this information as a point on the graph.

c) Use the graph to identify two different ages for which drivers have the same number of accidents. Use the equation for f to find the number of accidents for drivers at each of these ages.

How to Tell if a Graph Represents a Function: the Vertical Line Test

In general, for y to be a function of *x*, each value of *x* must be associated with exactly one value of *y*. Let us think of what this requirement means graphically. In order for a graph to represent a function, each *x*-value must correspond to exactly one *y*-value. This means that the graph must not intersect any vertical line at more than one point. Otherwise, the curve would contain two points with different *y*-values but the same *x*-value.

The vertical line test:
If any vertical line intersects a graph in more than one point, then the graph does not represent a function.

Exercise 3 (textbook 2.1)

Use the vertical line test to identify the graphs in which *y* is a function of *x*. For those graph, identify the domain and range.

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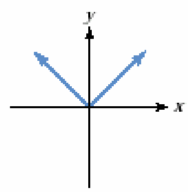

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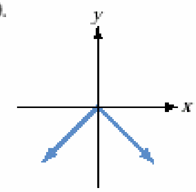

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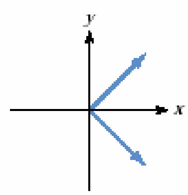

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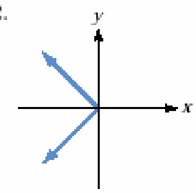

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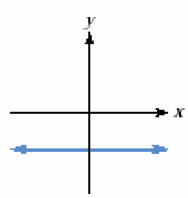

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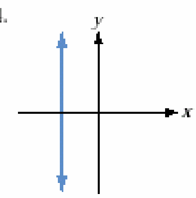

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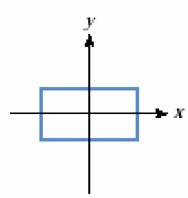

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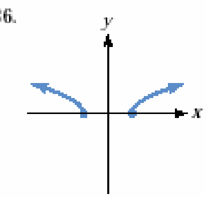


Exercise 4 a) Label the axes for a sketch of a problem, which says, "Sketch a graph of the cost of manufacturing q items…"

b) Label the axes for a problem, which says, "Graph the pressure, p, of a gas as a function of its volume, v, where p is in pounds per square inch and v is in cubic inches."

c) Label the axes for a problem which asks you to "Graph D in terms of y…"

Functions Defined by Equations

Exercise 5 Recall from geometry that if we know the radius of a circle, we can find its area. If we let $A = q(r)$ represent the area of a circle as a function of its radius, then a formula for $q(r)$ is

$$A = q(r) = pr^2.$$

Use the above formula, where r is in cm, to evaluate $q(10)$ and $q(20)$. Explain what your results tell you about circles.

Exercise 6 (textbook 2.1)

18. $f(x) = \dfrac{3x - 1}{x - 5}$

a. $f(0)$ b. $f(3)$ c. $f(-3)$

d. $f(10)$ e. $f(a + h)$

f. Why must 5 be excluded from the domain of f?

Exercise 7 Suppose $v(t) = t^2 - 2t$ gives the velocity, in ft/sec, of an object at time *t*, in seconds.

a) Is *v* a function of *t*? Explain.

b) Which variable is independent and which one is dependent?

c) What is $v(0)$ and what does it represent?

d) What is $v(3)$ and what does it represent?

Exercise 8 (textbook 2.1)

65. $f(x) = \begin{cases} 3x + 5 & \text{if } x < 0 \\ 4x + 7 & \text{if } x \geq 0 \end{cases}$

a. $f(-2)$ b. $f(0)$

c. $f(3)$ d. $f(-100) + f(100)$

The Algebra of Functions

Two functions *f* and *g* can be combined to form new functions $f + g,\ f - g,\ fg, \dfrac{f}{g}$ in a manner similar to the way we add, subtract, multiply and divide real numbers.

Definition

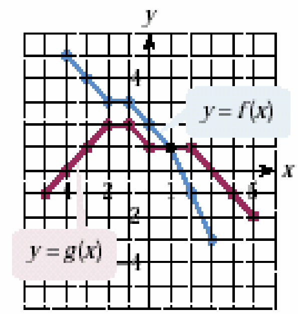
Let *f* and *g* be two functions. Let D_f be the domain of *f* and D_g the domain of *g*. Then:

- $(f+g)(x) = f(x) + g(x)$ and the domain of $f+g$ is $D_f \cap D_g$ (all real numbers that are common to the domain of *f* and the domain if *g*.)
- $(f-g)(x) = f(x) - g(x)$ and the domain of $f-g$ is $D_f \cap D_g$
- $(fg)(x) = f(x) \cdot g(x)$ and the domain of fg is $D_f \cap D_g$
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ and the domain of $\dfrac{f}{g}$ is the set of all real numbers that are common to the domain of *f* and the domain of *g* such that $g(x) \neq 0$

Exercise 9 $f(x) = x^2 + 4x$ and $g(x) = 2 - x$. Find $(f+g)(x), (f+g)(3), (fg)(1)$.

Exercise 10 (textbook 2.2)

Use the graphs of f and g to solve Exercises 49-56.



49. Find $(f + g)(-3)$. **50.** Find $(g - f)(-2)$.

51. Find $(fg)(2)$. **52.** Find $\left(\frac{g}{f}\right)(3)$.

53. Find the domain of $f + g$.

54. Find the domain of $\frac{f}{g}$.

Exercise 11 (textbook 2.2)

69. A company that sells radios has yearly fixed costs of \$600,000. It costs the company \$45 to produce each radio. Each radio will sell for \$65. The company's costs and revenue are modeled by the following functions:

$C(x) = 600{,}000 + 45x$ This function models the company's costs.

$R(x) = 65x$ This function models the company's revenue.

Find and interpret $(R - C)(20{,}000)$, $(R - C)(30{,}000)$, and $(R - C)(40{,}000)$.