2.1 Introduction to Functions
2.2 Graphs of Functions
2.3 The Algebra of Functions

The word *function*, used casually, expresses the notion of dependence.

- For example, a person might say that election results are a function of the economy, meaning that the winner of an election is determined by how the economy is doing.
- Another person may claim that car sales are a function of the weather, meaning that the number of cars sold on a given day is affected by the weather.

In nearly every physical phenomenon we observe that one quantity depends on another. For example, your height depends on your age, the temperature depends on the date, the cost of mailing a package depends on its weight. We use the term function to describe this dependence of one quantity on another. That is, we say:

- Height is a function of age.
- Temperature is a function of date.
- Cost of mailing a package is a function of weight.

Definition of Function

A function is a rule. To talk about a function , we need to give it a name. We will use letters such as *f*, *g*, *h*,… to represent functions.
For examples, we can use the letter *f* to represent a rule as follows:
"*f*" is the rule "square the number"
When we write $f(2)$, we mean "apply the rule *f* to the number 2". Applying the rule gives $f(2) = 2^2 = 4$. Similarly, $f(3) = 3^2 = 9$, and in general $f(x) = x^2$.

<u>Definition 1</u>

> **A function** is a rule that assigns to each element x in a set A exactly one element , called $f(x)$, in a set B.

<u>Notes</u>:

- We usually consider functions for which the sets A and B are sets of real numbers.
- The symbol $f(x)$ is read "f of x" or "*f* at *x*" and is called the **value of *f* at *x***, or the **image of *x* under *f*.**
- The set **A** is called the **domain** of the function.
- The **range** of a function *f* is the set of all possible values of $f(x)$ as x varies throughout the domain.

TABLE 1
Millions of CDs sold, by year

Year	Sales (millions)
1982	0
1983	0.8
1984	5.8
1985	23
1986	53
1987	102

Example 1 In the early 1980s, the recording industry introduced the compact disc (CD) as an alternative to vinyl long playing records (LPs). Table 1 gives the number of units (in millions) of CDs sold for the years 1982 through 1987. The year uniquely determines the number of CDs sold. Thus, we say that the number of CDs sold is a function of the year. We may also say that the number of CDs sold depends on the year.

Definition 2

A function is a relationship between two variables: **independent variable (input)** and **dependent variable (output)** that assigns to each independent variable a unique value of the dependent variable.

Function Notation

There is a convenient notation we use when discussing functions. First, we choose a name for the particular function, let's say *f*. That is, *f* is the name of the relationship between the two variables. Let's say *t* (the year) and *S* (the CD's sold) from Example 1. Then we can write

$$S = f(t)$$

which means " *S* is a function of *t*, and *f* is the name of the function."

Domain and Range

If $y = f(x)$, then

- the **domain** of *f* is the set of values for the independent variable *x*
- the **range** of *f* is the set of values for the dependent variable *y*

Functions Defined by Tables

TABLE 2

Cost of merchandise (M)	Shipping Charges (C)
\$0.01 – 10.00	\$2.50
10.01 – 20.00	3.75
20.01 – 30.00	4.85
30.01 – 50.00	5.95
50.01 – 75.00	7.95
Over 75.00	8.95

Example 2 a) Is *C* a function of *M*? Explain.

b) Is *M* a function of *C*? Explain.

c) If $C = f(M)$, find $f(3)$. What does it mean?

d) Solve $f(M) = 7.95$. What does it mean?

Example 3 Tables 3a, 3b, and 3c represent the relationship between the button number, *N*, which you push, and the snack, *S*, delivered by three different vending machines.

Table 3a
Vending Machine #1

N	*S*
1	m&ms
2	pretzels
3	dried fruit
4	Hersheys
5	fat free cookies
6	Snickers

Table 3b
Vending Machine #2

N	*S*
1	m&ms or dried fruit
2	Pretzels or Hersheys
3	Snickers or fat free cookies

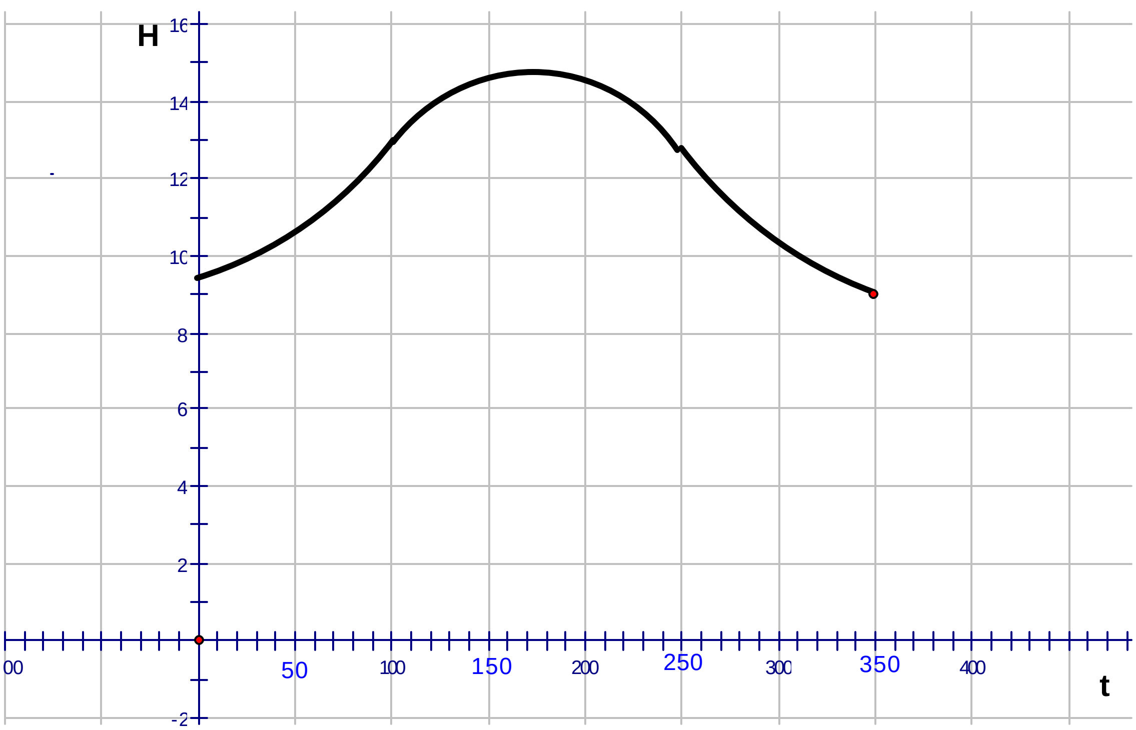
Table 3c
Vending Machine #3

N	*S*
1	m&ms
2	m&ms
3	dried fruit
4	Hersheys
5	Hersheys
6	fat free cookies
7	Snickers
8	Snickers

One of these vending machines is not a good one to use, because *S* is not a function of *N*. Which one? Explain why this makes it a bad machine to use.

Functions Defined By Graphs

Example 4 Figure 4 shows the number of hours, *H*, that the sun is above the horizon in Peoria, Illinois, on day *t*, where January 1 corresponds to $t = 0$.



a) Which variable is independent, and which is dependent?
b) Approximately how many hours of sunlight are there in Peoria on day 150?
c) On which days are there 12 hours of sunlight?
d) What are the maximum and minimum values of *H*, and when do these values occur?

Exercise 1 (textbook 2.2 - # 19 – 24)

In Exercises 19–24, use the graph of f to find each indicated function value.

19. $f(-2)$

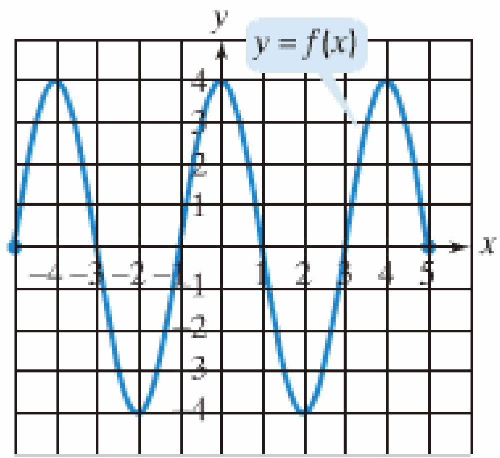
20. $f(2)$

21. $f(4)$

22. $f(-4)$

23. $f(-3)$

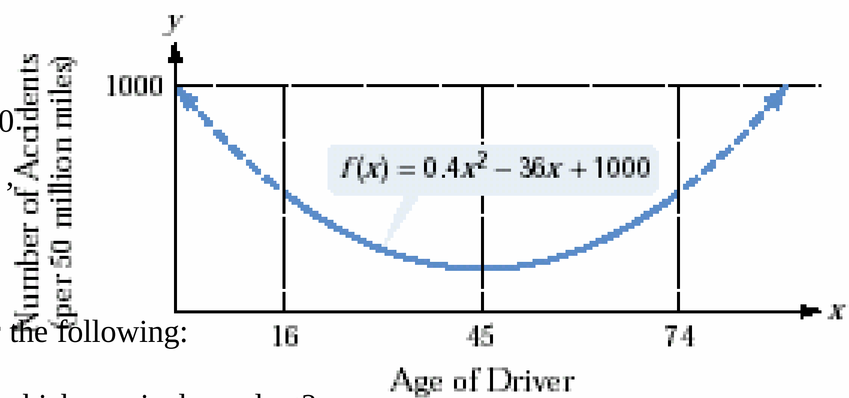
24. $f(-1)$



a) Is *y* a function of *x*? Explain.
b) Find the domain and range of *f*.

Exercise 2 (textbook 2.2 - #45, 48)

The function $f(x) = 0.4x^2 - 36x + 1000$
Models the number of accidents, $f(x)$, per 50 million miles driven as a function of a driver's age, *x*, in years, where *x* includes drivers from ages 16 through 74, inclusive. Answer the following:



a) Which variable is independent and which one is dependent?

b) Find and interpret $f(20)$. Identify this information as a point on the graph.

c) Use the graph to identify two different ages for which drivers have the same number of accidents. Use the equation for *f* to find the number of accidents for drivers at each of these ages.

Exercise 3 a) Label the axes for a sketch of a problem, which says, "Sketch a graph of the cost of manufacturing *q* items…"

b) Label the axes for a problem, which says, "Graph the pressure, *p*, of a gas as a function of its volume, *v*, where *p* is in pounds per square inch and *v* is in cubic inches."

c) Label the axes for a problem which asks you to "Graph *D* in terms of *y*…"

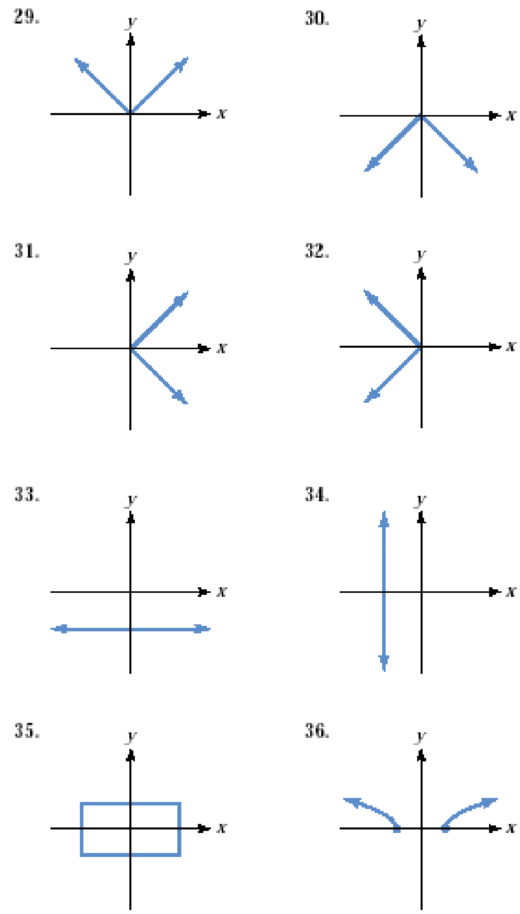
How to Tell if a Graph Represents a Function: the Vertical Line Test

In general, for y to be a function of x, each value of x must be associated with exactly one value of y. Let us think of what this requirement means graphically. In order for a graph to represent a function, each x-value must correspond to exactly one y-value. This means that the graph must not intersect any vertical line at more than one point. Otherwise, the curve would contain two points with different y-values but the same x-value.

> **The vertical line test:**
>
> A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

Exercise 4 (textbook 2.2 - # 11 – 18)

Use the vertical line test to identify the graphs in which y is a function of x. For those graph, identify the domain and range.



Functions Defined by Equations

Exercise 5 (textbook 2.1 - # 20)

Let $f(x) = \dfrac{3x-1}{x-5}$. Find the following:

a) $f(0)$ b) $f(3)$ c) $f(t)$ d) $f(a+h)$

e) What is the domain of this function?

Exercise 6 Suppose $v(t) = t^2 - 2t$ gives the velocity, in ft/sec, of an object at time *t*, in seconds.

a) Is *v* a function of *t*? Explain.

b) Which variable is independent and which one is dependent?

c) What is $v(0)$ and what does it represent?

d) What is $v(5)$ and what does it represent?

Exercise 7 (textbook 2.1- #31)

$f(x) = \begin{cases} 3x+5, & x<0 \\ 4x+7, & x \geq 0 \end{cases}$ Find the following: $f(-2), f(0), f(3), f(-100)+f(100)$.

What is the domain of the function? (This is an example of a piecewise defined function.)

Exercise 8 (textbook 2.1 - #57)

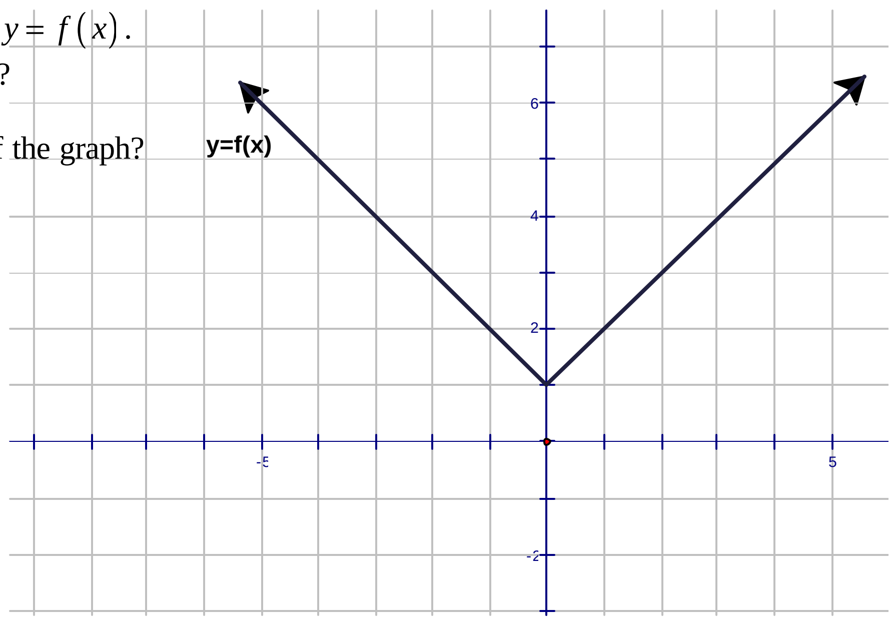
The graph shown represents $y = f(x)$.

a) Is *y* a function of *x*? Why?

b) What are the intercepts of the graph?

c) Find $f(2)$.

d) Solve $f(x) = 4$



Exercise 9 A function f is defined by the formula $f(x) = x^2 + 4$.

a) Express in words how *f* acts on the input *x* to produce the output $f(x)$.

b) Evaluate $f(3), f(-2), f(\sqrt{5})$.

c) Find the domain of the function.

d) Find the intercepts of the graph (if any).

The Algebra of Functions

Two functions *f* and *g* can be combined to form new functions $f+g,\ f-g,\ fg, \dfrac{f}{g}$ in a manner similar to the way we add, subtract, multiply and divide real numbers.

<u>Definition</u>

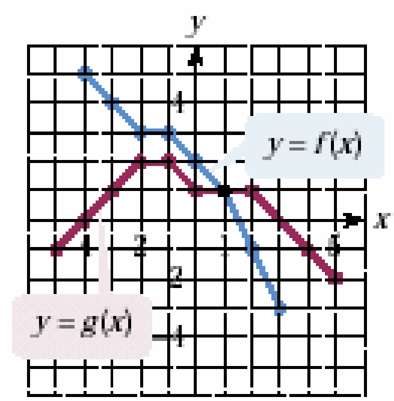
Let *f* and *g* be two functions. Let D_f be the domain of f and D_g the domain of *g*. Then:

- $(f+g)(x)=f(x)+g(x)$ and the domain of $f+g$ is $D_f \cap D_g$ (all real numbers that are common to the domain of *f* and the domain if *g*.)
- $(f-g)(x)=f(x)-g(x)$ and the domain of $f-g$ is $D_f \cap D_g$
- $(fg)(x)=f(x)\cdot g(x)$ and the domain of fg is $D_f \cap D_g$
- $\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$ and the domain of $\dfrac{f}{g}$ is the set of all real numbers that are common to the domain of f and the domain of *g* such that $g(x)\neq 0$

Exercise 10 $f(x)=x^2+4x$ and $g(x)=2-x$. Find $(f+g)(x), (f+g)(3), (fg)(1)$.

Exercise 11 (textbook 2.3 - #51 - 58)

Use the graphs of f and g to solve Exercises 49- 56.



49. Find $(f+g)(-3)$.

50. Find $(g-f)(-2)$.

51. Find $(fg)(2)$.

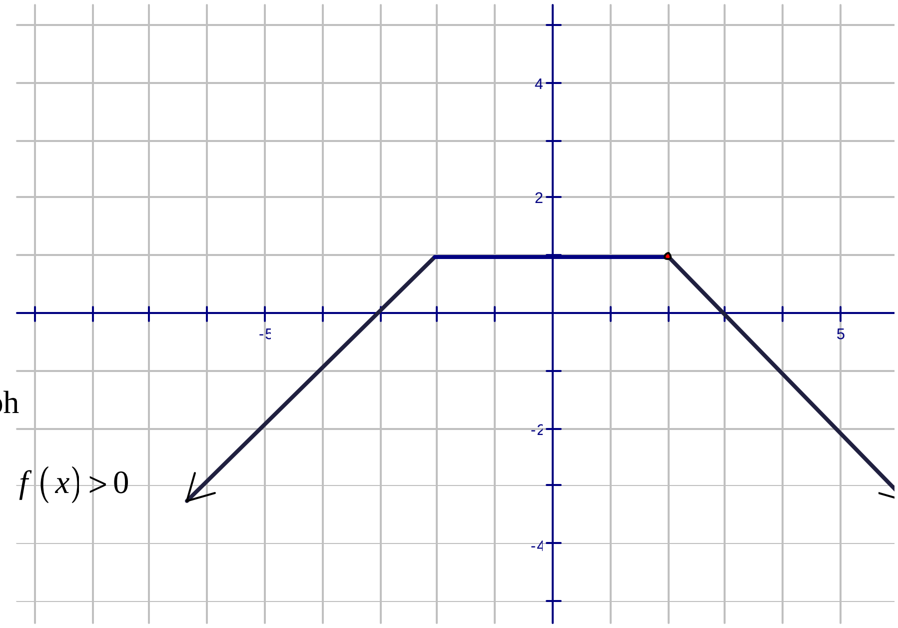
52. Find $\left(\dfrac{g}{f}\right)(3)$.

53. Find the domain of $f+g$.

54. Find the domain of $\dfrac{f}{g}$.

Exercise 12 (textbook 2.2 - # 42)

Use the graph of f to determine each of the following. Where applicable, use set-builder notation.



a) the domain of f

b) the range of f

c) $f(-4)$

d) the values of x for which $f(x) = -3$

e) the intercepts of the graph

f) the values of x for which $f(x) > 0$

More practice

Exercise 13 a) Let $n = f(A)$ be the amount of paint (in gallons) needed to cover an area of $A\ ft^2$. Explain in words what the following statement tells you about painting houses:

$$f(10{,}000) = 40 .$$

b) Let $F = g(t)$ be the number of foxes in a national park as a function of t, the number of months since January 1. What does $g(9)$ indicate?

Exercise 14 The graph in the figure defines $f(x)$.

Use it to estimate:

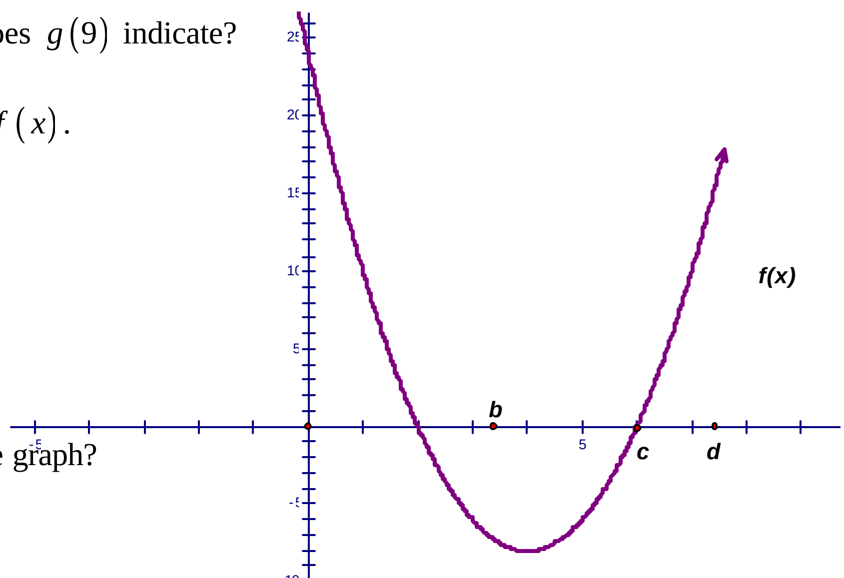
a) $f(0)$ b) $f(1)$

c) $f(b)$ d) $f(c)$

e) $f(d)$

f) Solve $f(x) = 10$ for x.

g) What are the intercepts of the graph?



Exercise 15 Let $f(x) = \sqrt{x^2 + 16} - 5$.

a) Find $f(0)$. What does it represent?

b) For what values of x is $f(x)$ zero? What does this represent?

c) Find $f(3)$.

Exercise 16 a) Algebraically find the domain of $y = \frac{1}{\sqrt{x-4}}$.

b) Algebraically find the range of $y = 2 + \frac{1}{x}$.

Exercise 17 Let $g(x) = \frac{x^2+1}{5+x}$. Evaluate the following expressions.

a) $g(3)$ b) $g(-1)$ c) $g(a)$

d) $g(a-2)$ e) $g(a)-2$ f) $g(a)-g(2)$.

Exercise 18 (textbook 2.3 - # 2, 3)

Find the domain of each function:

a) $f(x) = 4x+7$

b) $g(x) = \frac{1}{x+4}$

c) $h(x) = \frac{1}{\sqrt{x+7}} + \frac{1}{x-9}$

Exercise 19 (textbook page 135 - # 15, 17)

Let $f(x) = x^2 - 3x + 8$ and $g(x) = -2x - 5$.

a) Find the domain of each function.
b) Find $f(0) + g(-10)$.
c) Find $f(a) + g(a+3)$
d) Find $\frac{g(x+h) - g(x)}{h}$, where $h \neq 0$.
e) Find $\frac{f(a+h) - f(a)}{h}$, where $h \neq 0$.

Exercise 20 A cell phone plan costs $39 a month. The plan includes 400 free minutes and charges 20 cents for each additional minute of usage.

a) Write a piecewise defined function to represent the monthly charges as a function of the number of minutes used, x.
b) Find the cost of using the cell phone for 100 minutes. Write the answer using function notation.
c) Find the cost of using the cell phone for 480 minutes. Write the answer using function notation.