6.1 Rational Functions

Example : The Average Cost of Producing a Therapeutic Drug

Suppose a pharmaceutical company wants to begin production of a new therapeutic drug. The total cost *C*, in thousands of dollars, of making q grams of the drug is given by the linear function

$$C(q) = 2500 + 2q$$

a) Find $C(0)$ and its meaning.

b) Find the slope of the line and its meaning.

The fixed cost of $2.5 million is very large compared to the unit cost of $2000/gram. This means that it would be impractical for the company to make a small amount of the drug.

c) Find the cost of making only 10 grams of the drug.

However, as larger and larger quantities of the drug are manufactured, the initial outlay of $2.5 million will seem less significant. The fixed cost will "average out" over large number of units.

d) Find the average cost of producing a gram of the drug if the company makes 10,000 grams of the drug.

To help us think about the average cost of producing q units of the drug, we define the average cost function a as follows:

$$a(q) = \begin{pmatrix} \text{average cost of} \\ \text{producing q units} \end{pmatrix} = \frac{\begin{pmatrix} \text{total cost of producing} \\ \text{q grams} \end{pmatrix}}{\begin{pmatrix} \text{number of grams} \\ \text{produced} \end{pmatrix}} = \frac{C(q)}{q} = \frac{2500+2q}{q}$$

The average cost function $a(q)$ gives the cost per gram the company spends to produce q grams.

What is a Rational Function?

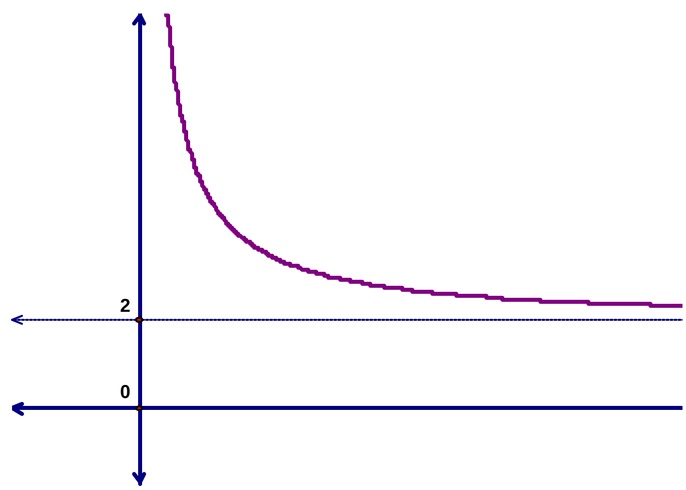
The function a is an example of a rational function. A rational function is a function given by the ratio of two polynomials.

Give some examples of rational functions:

The figure gives the graph of $y = a(q)$ for $q > 0$.

a) What is the domain of the function?

b) What is the behavior of the graph when $q \to \infty$, that is for larger and larger q? What does it mean?



c) What is the behavior of the graph when $q \to 0$, that is for smaller and smaller q ? What does it mean?

The graph of $y = a(q)$ has two asymptotes:

- A vertical asymptote at $q = 0$
- A horizontal asymptote at $y = 2$.

The horizontal asymptote of a reflects the fact that for large values of q, the value of $a(q)$ draws close to 2. This is reasonable: as more and more of the drug is produced, the initial $2.5 million expenditure grows increasingly less significant, whereas the unit cost of $2000 per gram remains unchanged. Thus, as more and more of the drug is produced, the average cost gets closer and closer to $2000 per gram. Complete the Table that gives the total cost $C(q)$ and the average cost $a(q)$ for producing various quantities of the drug. What happens with the values of $a(q)$ as q grows large? ____

q	C(q)=2500+2q	a(q)=C(q)/q
10,000		
30,000		
50,000		
100,000		
500,000		

On the other hand, the vertical asymptote tells us that the average cost per gram will be very large if only a small amount of the drug is made. As q approaches zero, the average cost $a(q)$ becomes extremely large. This is because the $2.5 million initial investment must be averaged out over very few units. For example , as we have seen, to produce only 10 grams costs $252,000 per gram.

Notations:

$x \to \infty$ x approaches infinity (x increases without bound)

$x \to -\infty$ x approaches negative infinity (x decreases without bound)

$x \to a^+$ x approaches a from the right

$x \to a^-$ x approaches a from the left

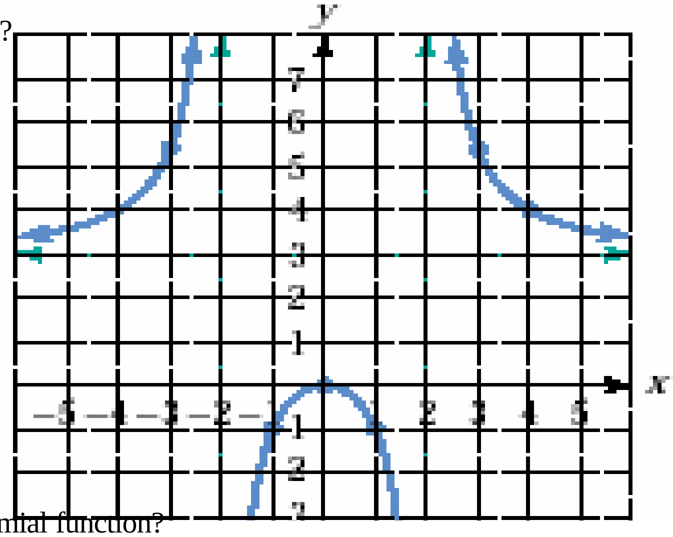
Definition

The line $\boxed{x = a}$ is a **vertical asymptote** for the graph of $f(x)$ if , when $x \to a$, $y \to \pm\infty$.

The line $\boxed{y = b}$ is a **horizontal asymptote** for the graph of $f(x)$ if, when $x \to \pm\infty$, $y \to b$.

Exercise 1: Textbook # 17 – 26

The graph of a rational function, f, is shown in the figure. Answer all questions:



a) What is the domain of the function? What is the range?

b) Find $f(4)$ and $f(1)$.

c) What are the vertical asymptotes of the graph?

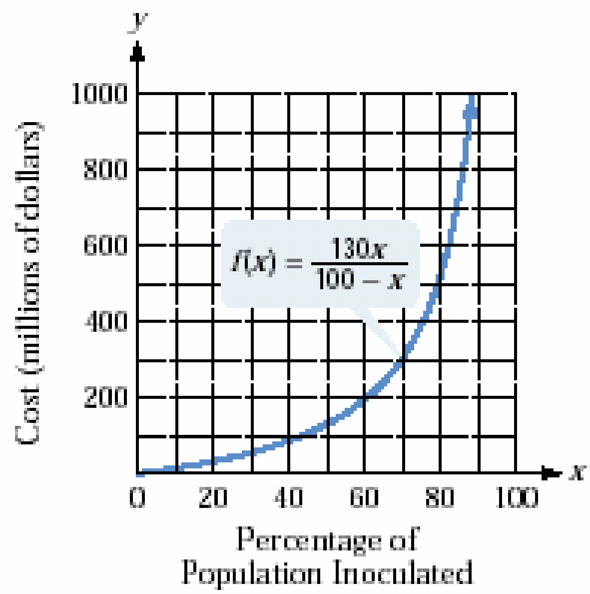
d) What is the horizontal asymptote?

e) How can you tell that this is not the graph of a polynomial function?

f) List two real numbers that are not function values of f.

Exercise #2: Textbook #105 – 108

The rational function $f(x) = \dfrac{130x}{100 - x}$ models the cost, $f(x)$, in millions, to inoculate x% of the population against a particular strain of flu. Answer the following:



a) What is the domain of the function? What is the range?
What is the meaning of the domain of the function?

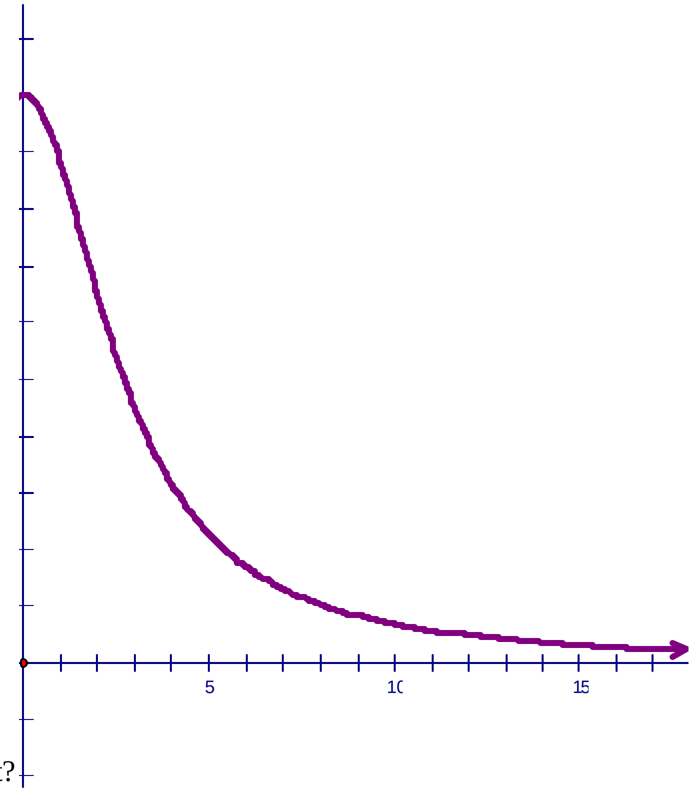
b) What happens to the cost as x approaches 100%?
How is this shown by the graph? Explain what it means.

c) Find and interpret $f(60)$. Identify your solution as a point on the graph.

Exercise #3

The rational function $P(x) = \dfrac{72{,}900}{100x^2 + 729}$ models the percentage of people in the U.S., $P(x)$, with x years of education who are unemployed. Answer the following:

a) What is the domain of the function?
What is the range?



b) Find and interpret $P(10)$.

c) Describe the end behavior of the graph. Is there an education level that leads to guaranteed employment? How is this indicated by the graph?

d) What happens when x approaches 0? What does it mean?