6.1 Rational Functions

Example: The Average Cost of Producing a Therapeutic Drug

Suppose a pharmaceutical company wants to begin production of a new therapeutic drug. The total cost C, in thousands of dollars, of making q grams of the drug is given by the linear function

$$C(q) = 2500 + 2q$$

a) Find $C(0)$ and its meaning.

$C(0) = 2500$ OR \$2,500,000. It tells us that the company must make an initial \$2,500,000 investment before it ~~starts~~ making the drug. This quantity is known as the fixed cost. It represents the cost for research, testing, and equipment

b) Find the slope of the line and its meaning.

$m = 2$ OR 2000 \$/gram $\left(m = \dfrac{\Delta C}{\Delta q}\right.$, so the units are $\left.\dfrac{\text{dollars}}{\text{gram}}\right)$

It tells that each gram of the drug costs an extra \$2000 to make. This quantity is known as the unit cost

The fixed cost of \$2.5 million is very large compared to the unit cost of \$2000/gram. This means that it would be impractical for the company to make a small amount of the drug.

c) Find the cost of making only 10 grams of the drug.

$C(10) = 2500 + 2(10) = 2520$

10 grams would cost \$2,520,000 to make; that is an average cost of $\dfrac{2{,}520{,}000\,\$}{10 \text{ grams}} = 252{,}000$ \$/gram

However, as larger and larger quantities of the drug are manufactured, the initial outlay of \$2.5 million will seem less significant. The fixed cost will "average out" over large number of units.

d) Find the average cost of producing a gram of the drug if the company makes 10,000 grams of the drug.

$$\frac{\left(\begin{array}{c}\text{total cost of producing} \\ \text{10,000 grams}\end{array}\right)}{\text{10,000 grams}} = \frac{2500 + 2(10{,}000)}{10{,}000} = 2.25 \text{ OR}$$

\$2250 per gram of drug produced

To help us think about the average cost of producing q units of the drug, we define the average cost function a as follows:

$$a(q) = \begin{pmatrix} \text{average cost of} \\ \text{producing q units} \end{pmatrix} = \frac{\begin{pmatrix} \text{total cost of producing} \\ \text{q grams} \end{pmatrix}}{\begin{pmatrix} \text{number of grams} \\ \text{produced} \end{pmatrix}} = \frac{C(q)}{q} = \frac{2500+2q}{q}$$

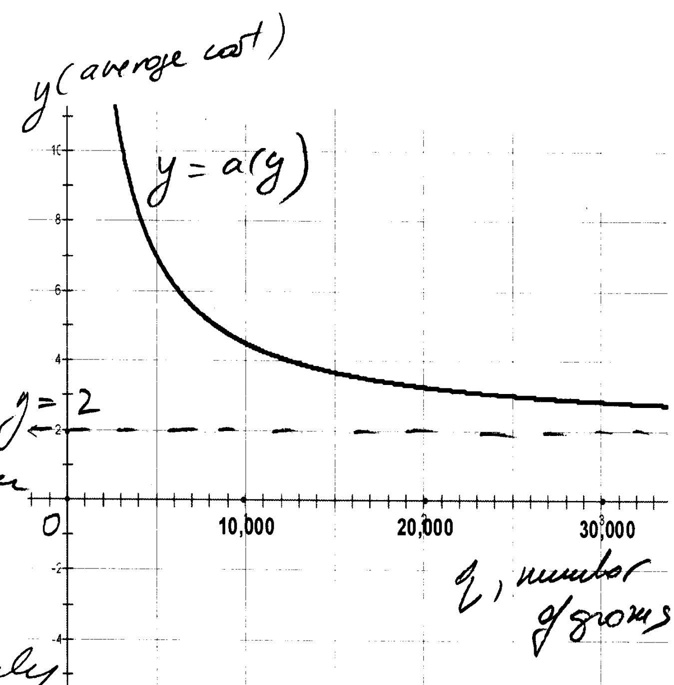
The average cost function $a(q)$ gives the cost per gram the company spends to produce q grams.

What is a Rational Function?

The function a is an example of a rational function. A rational function is a function given by the ratio of two polynomials.

Give some examples of rational functions:

The figure gives the graph of $y = a(q)$ for $q > 0$.



a) What is the domain of the function?

$q > 0$

b) What is the behavior of the graph when $q \to \infty$, that is for larger and larger q? What does it mean?

The graph approaches the horizontal line $y = 2$. As more of the drug is produced, the average cost gets closer to $2000/gram

c) What is the behavior of the graph when $q \to 0$, that is for smaller and smaller q ? What does it mean?

The graph rises; the average cost per gram will be very large if only a small amount of drug is produced

The graph of $y = a(q)$ has two asymptotes:

- A vertical asymptote at $q = 0$
- A horizontal asymptote at $y = 2$.

The horizontal asymptote of a reflects the fact that for large values of q, the value of $a(q)$ draws close to 2. This is reasonable: as more and more of the drug is produced, the initial \$2.5 million expenditure grows increasingly less significant, whereas the unit cost of \$2000 per gram remains unchanged. Thus, as more and more of the drug is produced, the average cost gets closer and closer to \$2000 per gram. Complete the Table that gives the total cost $C(q)$ and the average cost $a(q)$ for producing various quantities of the drug. What happens with the values of $a(q)$ as q grows large? a(q) approaches 2 $a(q) \to 2$ when $q \to \infty$

q	C(q)=2500+2q TOTAL COST	a(q)=C(q)/q AVERAGE
10,000	2500 + 20,000 = 22,500 \$	2.250 OR 2250 \$/gram
30,000	2500 + 60,000 = 62,500 \$	2.083 OR 2083 \$/gram
50,000	2500 + 100,000 = 102,500 \$	2.050 OR 2050 \$/gram
100,000	2500 + 200,000 = 202,500 \$	2.025 OR 2025 \$/gram
500,000	2500 + 1,000,000 = 1,002,500	2.005 OR 2005 \$/gram

On the other hand, the vertical asymptote tells us that the average cost per gram will be very large if only a small amount of the drug is made. As q approaches zero, the average cost $a(q)$ becomes extremely large. This is because the \$2.5 million initial investment must be averaged out over very few units. For example , as we have seen, to produce only 10 grams costs \$252,000 per gram.

Notations:

$x \to \infty$ — x approaches infinity (x increases without bound)

$x \to -\infty$ — x approaches negative infinity (x decreases without bound)

$x \to a^+$ — x approaches a from the right

$x \to a^-$ — x approaches a from the left

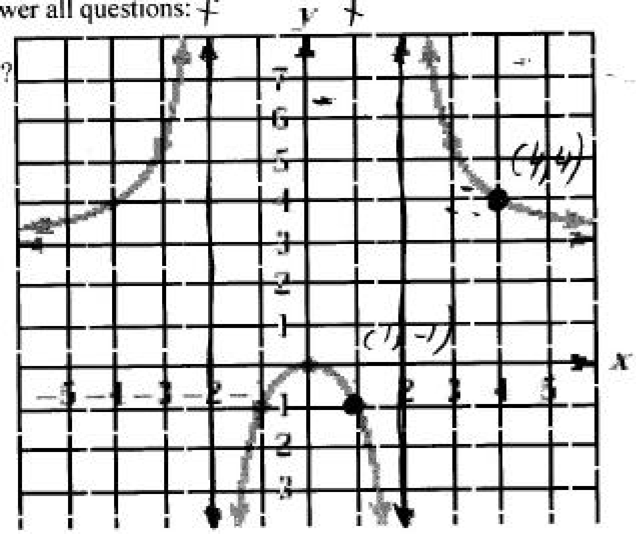
Definition

The line $\boxed{x=a}$ is a **vertical asymptote** for the graph of $f(x)$ if, when $x \to a$, $y \to \pm\infty$.

The line $\boxed{y=b}$ is a **horizontal asymptote** for the graph of $f(x)$ if, when $x \to \pm\infty$, $y \to b$.

Exercise 1: Textbook # 17 – 26

The graph of a rational function, f, is shown in the figure. Answer all questions:



a) What is the domain of the function? What is the range?

Domain: $x \in \mathbb{R} \setminus \{2, -2\}$

Range: $y \in (-\infty, 0] \cup (3, \infty)$

b) Find $f(4)$ and $f(1)$.

$f(4) = 4$ $f(1) = -1$

c) What are the vertical asymptotes of the graph?

$x = 2$ and $x = -2$

d) What is the horizontal asymptote?

$y = 3$

e) How can you tell that this is not the graph of a polynomial function?

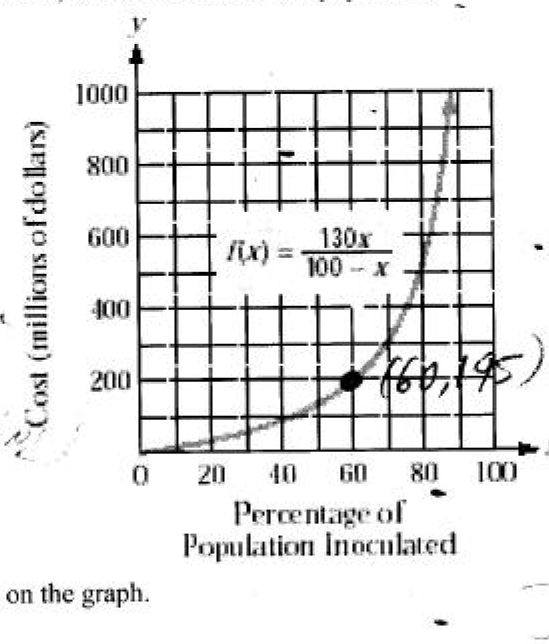
The graph is not continuous; it neither rises nor falls to the left or the right

f) List two real numbers that are not function values of *f*.

Any numbers $y \in (0, 3]$.

Exercise #2: Textbook #105 – 108

The rational function $f(x) = \dfrac{130x}{100 - x}$ models the cost, $f(x)$, in millions, to inoculate x% of the population against a particular strain of flu. Answer the following:



a) What is the domain of the function? What is the range?
What is the meaning of the domain of the function?

Condition: $x \neq 100$, $0 \le x < 100$

Domain: $\{x \mid 0 \le x < 100\}$

Range: $\{y \mid y \ge 0\} = [0, \infty)$

We cannot inoculate 100% of the population

b) What happens to the cost as x approaches 100%?
How is this shown by the graph? Explain what it means.

$x \to 100\%$, cost increasing

the more people are inoculated, the higher the cost

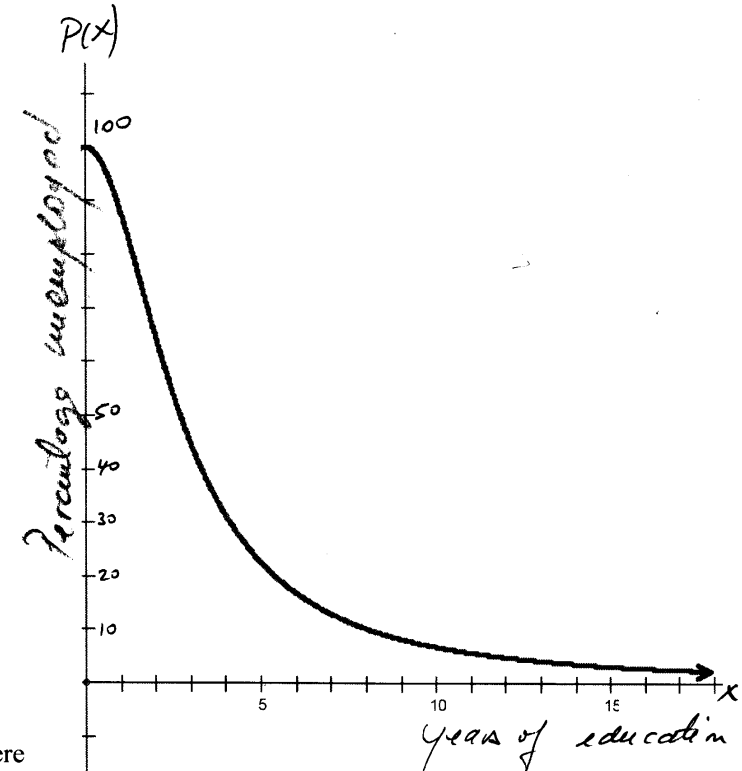
c) Find and interpret $f(60)$. Identify your solution as a point on the graph.

$f(60) = \dfrac{130(60)}{40} = 195$ million dollars

The cost to inoculate 60% of the population is \$195 million

Exercise #3: Textbook #109 – 112

The rational function $P(x) = \dfrac{72{,}900}{100x^2 + 729}$ models the percentage of people in the U.S., $P(x)$, with x years of education who are unemployed. Answer the following:



a) What is the domain of the function?
What is the range?

$x \geq 0$ 0 or more years of education

$P(x) \in [0, 100]$ % of people unemployed

b) Find and interpret $P(10)$.

$P(10) = \dfrac{72{,}900}{100(10)^2 + 729} \approx 7$

About 7% of the people with 10 years of education are unemployed

c) Describe the end behavior of the graph. Is there an education level that leads to guaranteed employment? How is this indicated by the graph?

As $x \to \infty$, $P(x) \to 0$

The more years of education, the smaller the % of unemployed people. The function values are approaching 0.
There is no education level that leads to guaranteed employment (the function never actually reaches 0)

d) What happens when x approaches 0? What does it mean?

$x \to 0$, $P(x) \to 100$

Actually, $P(0) = \dfrac{72{,}900}{100(0) + 729} = 100$

The unemployment rate approaches 100% as x approaches 0 (people with 0 years of education)