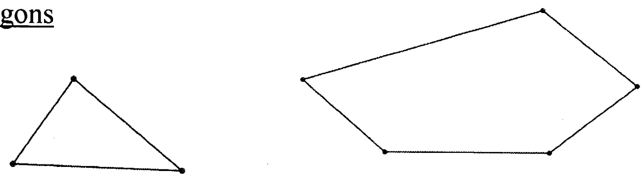
4.1 Parallelograms

Definition (3.3) A **polygon** is a closed figure whose sides are line segments that intersect only at endpoints. (*Polygon* is a word of Greek origin that means *many angles*; hence, it implies *many sides*).

Note: 1. We will be working only with **convex polygons,** polygons in which a line segment joining two points in the interior of the polygon has all its points in the interior of the polygon.
2. The angle measures of convex polygons are between 0° and 180°.

Examples of convex polygons



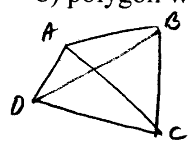
Definition (3.3) A **diagonal of a polygon** is a line segment that joins two nonconsecutive vertices.

Exercise #1 How many diagonals are in a

a) triangle

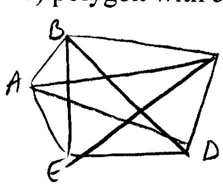
0 diagonals

b) polygon with 4 sides (quadrilateral)



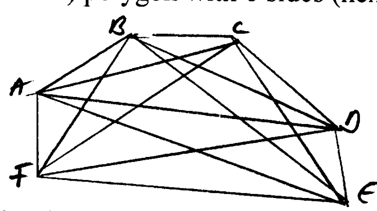
2 diagonals: $\overline{AC}$ and $\overline{BD}$

c) polygon with 5 sides (pentagon)



5 diagonals
$\overline{AC}$, $\overline{AD}$
$\overline{BD}$, $\overline{BE}$
$\overline{CE}$

d) polygon with 6 sides (hexagon)



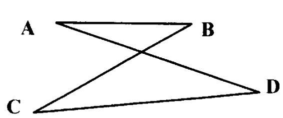
9 diagonals
$\overline{AC}$, $\overline{AD}$, $\overline{AE}$
$\overline{BD}$, $\overline{BE}$, $\overline{BF}$
$\overline{CE}$, $\overline{CF}$
$\overline{DF}$

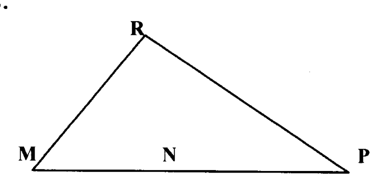
Definition (3.3) A **regular polygon** is a polygon with all its sides congruent and all its angles congruent.

Definition (4.1) A **quadrilateral** is a polygon that has four sides.

Note: - We will work only with quadrilaterals whose sides are coplanar.
- Special quadrilaterals (squares, rectangles, rhombuses, parallelograms, and trapezoids) occur in various practical circumstances, such as architectural design, construction materials, fabric design, and urban planning.

Important! ABCD is a quadrilateral iff all points are coplanar, no three of which are collinear, and each segment intersects exactly two others, one at each endpoint.
Therefore, the following figures are **not** quadrilaterals:

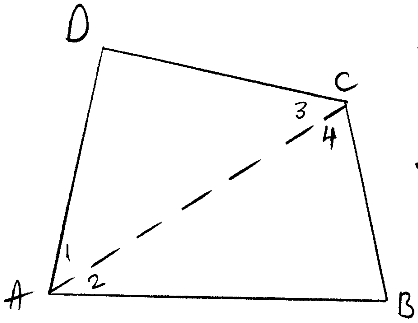




Property The sum of the interior angles of a quadrilateral is 360°.

Given $\underline{ABCD}$ quad.

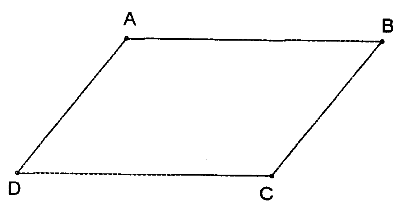
Prove $\underline{m\angle A + m\angle B + m\angle C + m\angle D = 360^\circ}$



Proof

Statements	Reasons
1. ABCD quad.	1. given
2. Draw $\overline{AC}$	2. 2 points determine a line
3. $m\angle 1 + m\angle 3 + m\angle D = 180^\circ$ ($\triangle ADC$) $m\angle 2 + m\angle 4 + m\angle B = 180^\circ$ ($\triangle ABC$)	3. Sum $\angle$'s in $\triangle = 180^\circ$
4. $(m\angle 1 + m\angle 3 + m\angle D) + (m\angle 2 + m\angle 4 + m\angle B) = 360^\circ$	4. + prop. of equality
5. $(m\angle 1 + m\angle 2) + (m\angle 3 + m\angle 4) + m\angle B + m\angle D = 360^\circ$	5. Associative prop. of +
6. $m\angle A = m\angle 1 + m\angle 2$ and $m\angle C = m\angle 3 + m\angle 4$	6. Angle-Addition Postulate
7. $m\angle A + m\angle B + m\angle C + m\angle D = 360^\circ$	7. Substitution

Definition (4.1) A **parallelogram** is a quadrilateral whose opposite sides are parallel.

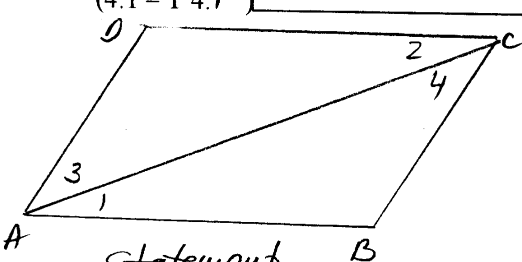


ABCD quadrilateral

ABCD parallelogram iff $\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{BC}$

The defining property for a parallelogram is that it si a quadrilateral whose opposite sides are parallel. Many other properties follow from this. The most significant feature of the figure is that for either pair of opposite sides, the other two sides and the diagonals are transversals. Thus, the theory of parallel lines and transversals may be used to prove properties of parallelograms. This theory and that for congruent triangles provide the needed tools for study of parallelograms.

Theorem 1 (4.1 – T 4.1) A diagonal of a parallelogram separates it into two congruent triangles.



Given: $\underline{\square ABCD}$ with $\overline{AC}$ diagonal

Prove: $\underline{\triangle ACD \cong \triangle CAB}$

Proof

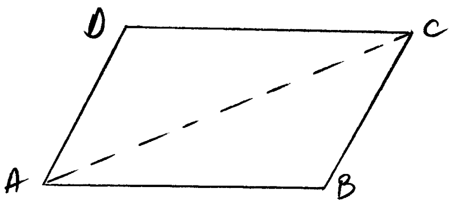
Statements	Reasons
1. $\square ABCD$	1. given
2. $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$	2. def. of parallelogram (opp. sides $\parallel$)
3. $\angle 1 \cong \angle 2$	3. Alt. int. $\angle$'s ($\overline{AB} \parallel \overline{CD}$ and transv. $\overline{AC}$)
4. $\angle 3 \cong \angle 4$	4. Alt. int. $\angle$'s ($\overline{AD} \parallel \overline{BC}$ and transv. $\overline{AC}$)
5. $\triangle ACD$, $\triangle CAB$: $\overline{AC} \cong \overline{AC}$, $\angle 3 \cong \angle 4$, $\angle 2 \cong \angle 1$	5. reflexive $\cong$; (4); (3)
6. $\triangle ACD \cong \triangle CAB$	6. ASA

Properties of Parallelograms

Corollaries
(4.1 –

1. The opposite sides of a parallelogram are congruent (opp sides ▱ $\cong$).
2. The opposite angles of a parallelogram are congruent (opp $\angle$'s ▱ $\cong$).
3. Any two consecutive angles of a parallelogram are supplementary (consec $\angle$'s ▱ supp).
4. The diagonals of a parallelogram bisect each other (diags ▱ bisect each other).

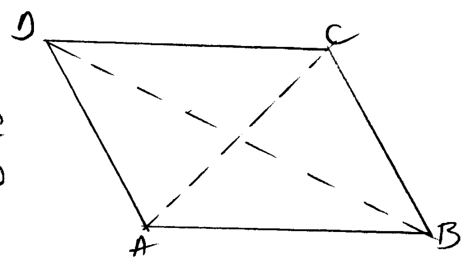
(1) Given ▱ABCD
Prove $\overline{AB} \cong \overline{CD}$
$\overline{AD} \cong \overline{BC}$



Proof of (1)

Statements	Reasons
1. ▱ ABCD	1. given
2. Draw $\overline{AC}$	2. 2 point determine a line
3. $\triangle ACD \cong \triangle CAB$	3. ▱, diag. forms $\cong \triangle$'s
4. $\overline{AD} \cong \overline{BC}$, $\overline{DC} \cong \overline{AB}$	4. CPCTC

(2) Given ▱ABCD
Prove $\angle A \cong \angle C$
$\angle B \cong \angle D$

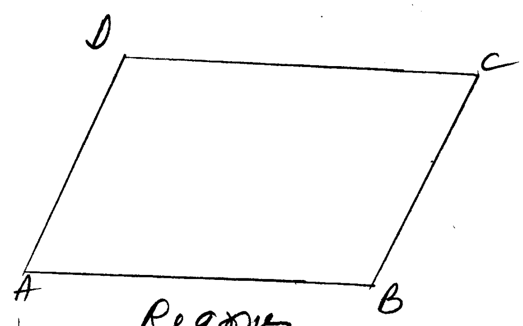


Proof of (2)

Statements	Reasons
1. ▱ ABCD	1. given
2. Draw $\overline{AC}$ and $\overline{BD}$	2. 2 point determine a line
3. $\triangle ACD \cong \triangle CAB$	3. In ▱, diag. form $\cong \triangle$'s
4. $\angle B \cong \angle D$	4. CPCTC
5. $\triangle ABD \cong CDB$	5. In ▱, diag. form $\cong \triangle$'s
6. $\angle A \cong \angle C$	6. CPCTC

(3) Given $\square ABCD$

Prove $\angle A$ supp. $\angle B$
$\angle B$ supp. $\angle C$
$\angle C$ supp. $\angle D$
$\angle D$ supp. $\angle A$

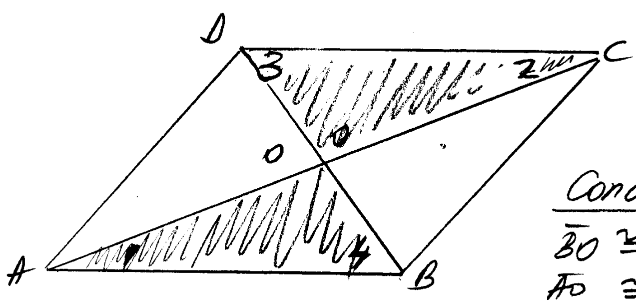


Proof of (3)

Statements	Reasons
1. $\square ABCD$	1. given
2. $\overline{AB} \parallel \overline{CD}$	2. def. of $\square$ ($\square$ iff. opp. sides $\parallel$)
3. $\angle A$ supp. $\angle D$ $\angle B$ supp. $\angle C$	3. same side int. $\angle$'s are supplementary $\overline{AB} \parallel \overline{CD}$ and transv. $\overline{AD}$ $\overline{AB} \parallel \overline{CD}$ and transv. $\overline{BC}$
4. $\overline{AD} \parallel \overline{BC}$	4. def. of $\square$ ($\square$ iff opp. sides $\parallel$)
5. $\angle A$ supp. $\angle B$ $\angle D$ supp. $\angle C$	5. same side int. $\angle$'s are supplementary $\overline{AD} \parallel \overline{BC}$ and transv. $\overline{AB}$ $\overline{AD} \parallel \overline{BC}$ and transv. $\overline{DC}$

(4) Given $\square ABCD$
$\overline{AC}$, $\overline{BD}$ diagonals

Prove $\overline{AC}$ bisects $\overline{BD}$
$\overline{BD}$ bisects $\overline{AC}$



Conditions
$\overline{BO} \cong \overline{DO}$
$\overline{AO} \cong \overline{CO}$

Proof of (4)

Statements	Reasons
1. $\square ABCD$, diag. $\overline{AC}$, $\overline{BD}$	1. given
2. $\overline{AB} \parallel \overline{DC}$	2. def. of $\square$ ($\square$ iff opp. sides $\parallel$)
3. $\angle 1 \cong \angle 2$ $\angle 4 \cong \angle 3$	3. alt. int. $\angle$'s $\overline{AB} \parallel \overline{DC}$ and transv. $\overline{AC}$ $\overline{AB} \parallel \overline{DC}$ and transv. $\overline{BD}$
4. $\overline{AB} \cong \overline{CD}$	4. opp. sides $\square$ $\cong$
5. $\triangle AOB$, $\triangle COD$: $\overline{AB} \cong \overline{CD}$, $\angle 1 \cong \angle 2$, $\angle 4 \cong \angle 3$	5. (4), (3), (3)
6. $\triangle AOB \cong \triangle COD$	6. ASA
7. $\overline{AO} \cong \overline{CO}$	7. CPCTC
8. $\overline{BD}$ bisects $\overline{AC}$	8. def. of bisector of a segment
9. $\overline{BO} \cong \overline{DO}$	9. CPCTC
10. $\overline{AC}$ bisects $\overline{BD}$	10. same as (8)

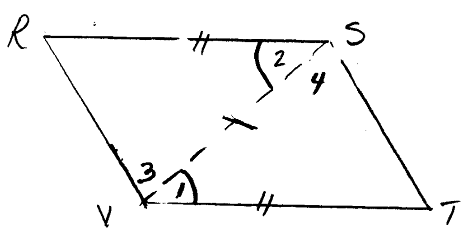
Summary: Properties of parallelograms

1. The opposite sides of a parallelogram are parallel.
2. Diagonal divides a parallelogram into two congruent triangles .
3. Any two consecutive angles of a parallelogram are supplementary.
4. The diagonals of a parallelogram bisect each other.
5. Opposite sides are congruent.
6. Opposite angles congruent.

When is a quadrilateral a parallelogram?

Theorem 3 (4.1 – T 4.6)

If two sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.



Given: $VTSR$ – quadrilateral
$\overline{RS} \parallel \overline{VT}$
$\overline{RS} \cong \overline{VT}$

Prove: $VTSR$ – parallelogram

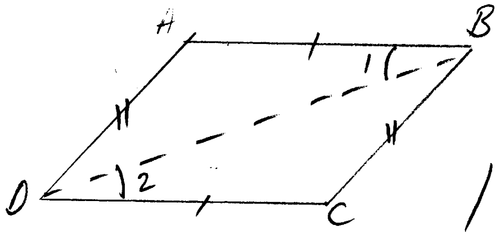
(condition: $\overline{VR} \parallel \overline{TS}$)

Proof.

Statements	Reasons.
1. $\overline{RS} \parallel \overline{VT}$; $\overline{RS} \cong \overline{VT}$	1. given
2. Draw $\overline{VS}$	2. 2 points determine a line
3. $\angle 1 \cong \angle 2$	3. alternate int. $\angle$'s ($\overline{RS} \parallel \overline{VT}$, $\overline{VS}$-transv)
4. $\triangle RSV$, $\triangle TVS$: $\overline{VS} \cong \overline{VS}$	4. reflexive prop. $\cong$
$\angle 2 \cong \angle 1$	(3) above
$\overline{RS} \cong \overline{VT}$	given
5. $\triangle RSV \cong \triangle TVS$	5. SAS
6. $\angle 3 \cong \angle 4$	6. CPCTC
7. $\overline{RV} \parallel \overline{ST}$	7. $\parallel$ iff. alternate int. $\angle$'s $\cong$ ($\overline{RV}$ and $\overline{ST}$ with transv. $\overline{VS}$)
8. $VTSR$ - parallelogram	8. definition of $\square$ ($\square$ iff opp. sides $\parallel$)

Theorem 4
(4.2 – T 4.11)

If both pairs of opposite sides of a quadrilateral are congruent, then it is a parallelogram.

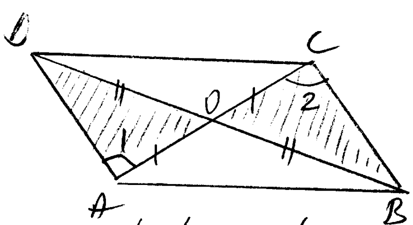


Given: ABCD quadrilateral
$\overline{AB} \cong \overline{DC}$
$\overline{AD} \cong \overline{BC}$
Prove: ABCD is a parallelogram
! (Condition: $\overline{AB} \parallel \overline{DC}$)

Statements	Reasons
1. ABCD - quadrilateral	1. given
2. Draw $\overline{BD}$	2. 2 points determine a line
3. $\triangle ABD$, $\triangle CDB$: $\overline{BD} \cong \overline{BD}$; $\overline{AB} \cong \overline{DC}$; $\overline{AD} \cong \overline{BC}$	3. reflexive $\cong$; given; given
4. $\triangle ABD \cong \triangle CDB$	4. SSS
5. $\angle 1 \cong \angle 2$	5. CPCTC
6. $\overline{AB} \parallel \overline{DC}$	6. $\parallel$ iff. alt. int $\angle$'s $\cong$ ($\overline{AB}$ and $\overline{DC}$ with transv. $\overline{BD}$)
7. ABCD = parallelogram	7. opp. sides $\parallel$ and $\cong$ ($\overline{AB} \overset{\parallel}{\cong} \overline{DC}$)

Theorem 5
(4.2 – T 4.12)

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

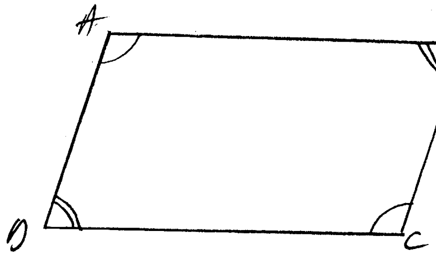


Given: ABCD - quadrilateral
$\overline{AC}$ and $\overline{BD}$ - diagonals
$\overline{AC}$ bisects $\overline{BD}$
$\overline{BD}$ bisects $\overline{AC}$
Prove: ABCD = parallelogram
! (Condition $\overline{AD} \overset{\parallel}{\cong} \overline{BC}$)

Statements	Reasons
1. ABCD - quad, $\overline{AC}$, $\overline{BD}$ diags.	1. given
2. $\overline{AC}$ bisects $\overline{BD}$	2. given
3. $\overline{BO} \cong \overline{DO}$	3. def. of bisector of a segment
4. $\overline{BD}$ bisects $\overline{AC}$	4. given
5. $\overline{AO} \cong \overline{CO}$	5. same as (3)
6. $\triangle AOD$, $\triangle COB$: $\overline{AO} \cong \overline{CO}$; $\overline{BO} \cong \overline{DO}$; $\angle AOD \cong \angle BOC$	6. (5); (3); vertical angles
7. $\triangle AOD \cong \triangle COB$	7. SAS
8. $\overline{AD} \cong \overline{BC}$ and $\angle 1 \cong \angle 2$	8. CPCTC
9. $\overline{AD} \parallel \overline{BC}$	9. $\parallel$ iff. alt. int $\angle$'s $\cong$ ($\overline{AD}$ and $\overline{BC}$ with transv. $\overline{AC}$)
10. ABCD $\square$	10. opp. sides $\parallel$ and $\cong$

Theorem 6
(4.1– T 4.6)

> If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram



Given: $ABCD$- quadrilateral
$\angle A \cong \angle C$
$\angle B \cong \angle D$

Prove: $ABCD$ = parallelogram

Condition: (opp. sides are $\parallel$)

Proof

1. $m\angle A + m\angle B + m\angle C + m\angle D = 360°$	1. Sum of $\angle$'s of quadrilateral $= 360$
2. $\angle A \cong \angle C$; $m\angle A = m\angle C$ $\angle B \cong \angle D$; $m\angle B = m\angle D$	2. given; def. $\cong \angle$'s
3. $m\angle A + m\angle B + m\angle A + m\angle B = 360°$ (1,2)	3. Substitution
4. $2m\angle A + 2m\angle B = 360°$	4. Simplifying (distributive)
5. $m\angle A + m\angle B = 180°$	5. Mult/division prop. of =
6. $\overline{AD} \parallel \overline{BC}$	6. $\parallel$ iff same side int. $\angle$'s $\cong$ ($\overline{AD}$, $\overline{BC}$ with $\overline{AB}$ transversal)
7. $m\angle A + m\angle D = 180°$ (2,5)	7. Substitution
8. $\overline{AB} \parallel \overline{CD}$	8. $\parallel$ iff. same side int. $\angle$'s $\cong$ ($\overline{AB}$ and $\overline{DC}$ with $\overline{AD}$ -transversal)
9. $ABCD$- parallelogram (6,8)	9. def of $\square$

Summary: Methods to prove a quadrilateral is a parallelogram

1. Show both pairs opposite sides are parallel.
2. Show both pairs of opposite sides are congruent.
3. Show both pairs of opposite angles are congruent.
4. Show one pair of opposite sides are congruent and parallel.
5. Show diagonal bisect each other.