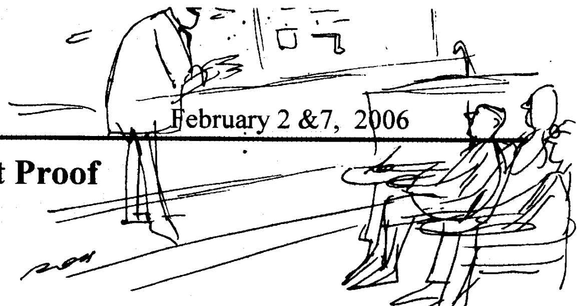
Section 3.1 Indirect Proof

A series of lessons in a subject that contradicted each other would make that subject very confusing. Yet, in reasoning deductively in geometry, it is sometimes helpful to try to arrive at contradictions deliberately!



"Don't take this lesson too seriously because tomorrow's lesson will contradict what you've learned today."

Proofs
- Direct - consist of a sequence of conditional statements such as $a \to b,\ b \to c,\ c \to d,\ d \to e$ that "link together" to yield a theorem, which in this case is $a \to e$.
- Indirect - involve reasoning to a contradiction.

Consider the following puzzle:

Emerson, Lake, and Palmer are different heights. Who is the tallest and who is the shortest if *only one* of the following statements is true?

1. Emerson is the tallest.
2. Lake is not the tallest.
3. Palmer is not the shortest.

One way of starting to figure this puzzle out would be to assume that the first statement is the one that is true, which would mean that the other two are false. This results in the following three statements:

1- TRUE
2- FALSE
3- FALSE
Then:

1. Emerson is the tallest.
2. Lake is the tallest
3. Palmer is the shortest

What is wrong? 1 and 2 contradict the info that they have different heights

Therefore, it is reasonable to conclude that, if there is nothing wrong with the puzzle itself, our assumption that the first statement is true must be wrong. Therefore, it is either the second or the third statement that is true.

On the assumption that the second statement is the true one, we get:

1- FALSE
2- TRUE
3- FALSE
Then:

1. Emerson is not the tallest
2. Lake is not the tallest
3. Palmer is the shortest

What is the contradiction? One of them has to be the tallest!

So, our assumption that the second statement is true is wrong.

That leaves just one more possibility if the puzzle has a solution at all; namely, that the third statement is the true one. We have:

1 - FALSE
2 - FALSE
3 - TRUE
Then:

1. Emerson is not the tallest.
2. Lake is the tallest.
3. Palmer is not the shortest

This time there is no contradiction. Evidently Lake is the tallest of the three and Emerson is the shortest.

The basic pattern in proving a theorem, say $P \rightarrow Q$, indirectly is to begin by assuming *not* Q ($\sim Q$). It is by reasoning from *this* statement that we hope to arrive at a contradiction. The statements Q and $\sim Q$ are called opposites of each other.

Exercise #1 — Write the opposites (negation) of the following statements:

a) Seven is a prime number. 7 is not a prime number

b) Mr. Spock does not like contradictions. Mr. Spock likes contradictions

c) A line contains at least two points. A line has less than 2 points
2 or more

To prove a theorem indirectly, we begin by assuming the opposite of its conclusion.

Exercise #2 — Write the opposite of the conclusion of each of the following theorems.

a) If a number is odd, its square is odd. The square of the number is not odd.

b) If two lines intersect, they intersect in no more than one point.
They intersect in more than one point.

c) In a plane, two lines perpendicular to a third line are parallel to each other.
The two lines are not parallel to each other.

Example of an indirect proof. Know: If it's poison ivy, then it has leaves in groups of three

It is sufficient to know that poison ivy has leaves in groups of three to prove that the plant in this photograph is not poison ivy.

a) With what assumption would we begin the proof?
Suppose the plant in the photograph is poison ivy.

b) What conclusion follows from this assumption?
The plant has leaves in groups of 3.

c) What does this conclusion contradict?
The fact that the plant doesn't have leaves in groups of 3

Since our initial assumption led to a contradiction, it must be false. In other words, the statement "The plant in this photograph is poison ivy" is false.

d) What statement, then, must be true? The plant is not poison ivy

Method of Indirect Proof

To prove the statement $P \rightarrow Q$ or to complete the proof problem of the form

Given: P

Prove: Q

where Q may be a negation, use the following steps:

1. Suppose that $\sim Q$ is true.
2. Reason from the supposition until you reach a contradiction.
3. Note that the supposition claiming that $\sim Q$ is true must be false and that Q must therefore be true.

Step 3 completes the proof.

Note that the contradiction that is discovered in an indirect proof often has the form $\sim P$. Thus the assumed statement $\sim Q$ has forced the conclusion $\sim P$, asserting that $\sim Q \rightarrow \sim P$ is true. Then the desired theorem $P \rightarrow Q$ (the contrapositive of $\sim Q \rightarrow \sim P$) is also true.

Exercise #3 | If a and b are positive numbers, then $\sqrt{a^2+b^2} \neq a+b$.

Given: $a>0,\ b>0$

Prove: $\sqrt{a^2+b^2} \neq a+b$

Proof:

Assume $\sqrt{a^2+b^2} = a+b$ $\quad /^2$

then $\left(\sqrt{a^2+b^2}\right)^2 = (a+b)^2$

$a^2+b^2 = a^2+2ab+b^2$

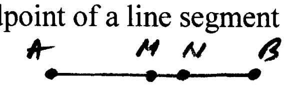
$2ab = 0 \Rightarrow$

$a=0$ OR $b=0$

Contradiction with given $a>0$ and $b>0$

Therefore, $\sqrt{a^2+b^2} \neq a+b$

Exercise #4 | The midpoint of a line segment is unique.



Given: $\overline{AB}$, $M =$ midpoint of $\overline{AB}$

Prove: $M =$ unique (the only one)

Proof:

Assume $M \neq$ unique

Then there is $N \neq M$, $N =$ midpoint

so $N \in \overline{AB}$, $AN = NB$

M-midpoint $\Rightarrow AM = \frac{1}{2}AB$

N-midpoint $\Rightarrow AN = \frac{1}{2}AB$ $\Big\} \Rightarrow AM = AN$

Also, $AN = AM + MN$

$\Big\} \Rightarrow MN = 0$ contradiction with $M \neq N \Rightarrow M =$ unique

3.1 Parallel Lines

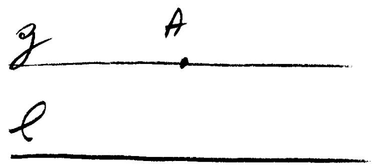
Definition **Parallel lines** are lines that lie in the same plane but do not intersect.

Postulate

> If two lines intersect, they intersect at one point.

Postulate (3.1 – P 3.1)

> For a given line and a given point not on the line, there is one and only one line through the given point parallel to the given line.



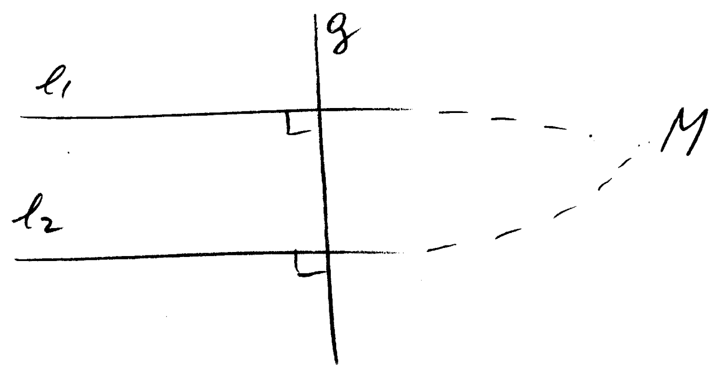
Given l · line
$A \notin l$
then there is only one ($\exists !$) line $g \parallel l$ such that $A \in g$

Theorem (3.1 – T 3.1)

> If two coplanar lines are each perpendicular to a third line, then these lines are parallel to each other.

Hypothesis: $\begin{cases} l_1 \perp g \\ l_2 \perp g \end{cases}$

Conclusion: $l_1 \parallel l_2$



Proof:

Assume $l_1 \nparallel l_2$

Then $l_1 \cap l_2 = M$

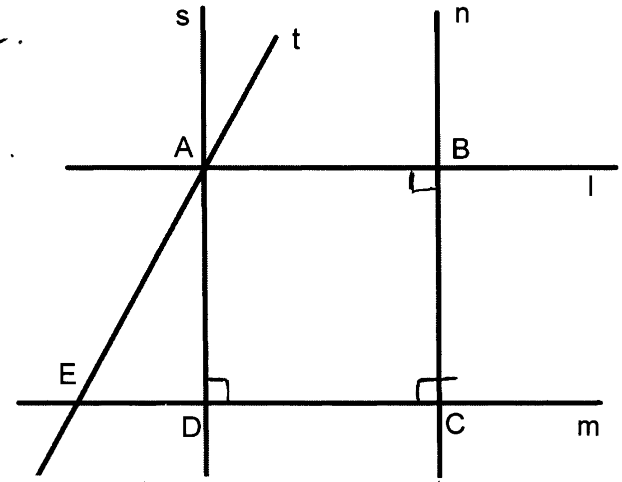
Given line g and point M, there are two lines through M perpendicular to g $\left(\begin{array}{l} l_1 \perp g \\ l_2 \perp g \\ l_1 \ni M,\ l_2 \ni M \end{array} \right)$

This contradicts the Postulate that says that there is only one line $\perp$ to a given line passing through a given point.

Therefore, the assumption $l_1 \nparallel l_2$ is incorrect.

Therefore, $l_1 \parallel l_2$.

<u>Exercise #5</u> Answer the following:
(3 1- # 5 - 14)



a) Is $l \parallel m$? yes, both are $\perp$ to n. (Theorem)

b) Is $s \parallel n$? yes, both are $\perp$ to m.

c) Is $\overline{AB} \parallel \overline{ED}$? yes, both $\perp$ n

d) Is $\overline{AD} \parallel \overline{EC}$? No; $\overleftrightarrow{AD} \cap \overleftrightarrow{EC} = \{D\}$

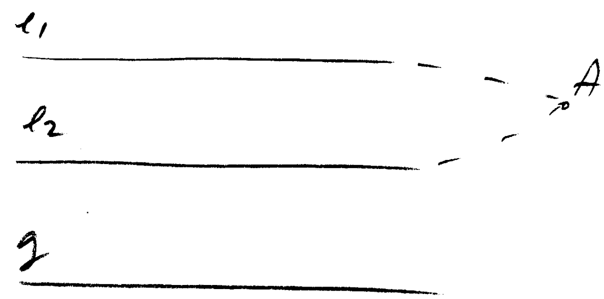
e) Can t be parallel to n?
No; through a given point (A) there is only one line parallel to a given line ($s \parallel n$)

f) Does there exist a line through E parallel to n?
Yes; Postulate (

g) How many lines through B are parallel to m?
One and only one (l)

h) Is $\overline{AB} \parallel \overline{EC}$?
yes, both $\perp$ n.

<u>Exercise #6</u> If two lines are parallel to a third line then they are parallel to each other.
(3.1 - # 15)



Given: $l_1 \parallel g$
$l_2 \parallel g$

Prove: $l_1 \parallel l_2$

<u>Proof</u>

Assume $l_1 \nparallel l_2$
Then $l_1 \cap l_2 = \{A\}$ $A \notin g$
Given A, and line g, there are two lines through A parallel to g.
Contradiction.
Therefore, $l_1 \parallel l_2$