Definition A **STATEMENT** is a group of words and symbols that can be classified collectively as true or false, but not both simultaneously.

Exercise #1 Which sentences are statements? If a sentence is a statement, classify it as true or false.

a) Where do you live? not a statement

b) $4+7 \neq 5$ statement; true

c) Washington was the first U.S president. statement; true

d) $x+3=7$ when $x=5$. statement; false

Note: We represent statements by letters such as P, Q, and R.

Definition The **NEGATION** of a given statement P makes a claim opposite that of the original statement. The negation of a true statement is false, and the negation of a false statement is true.

If P is a statement, $\sim P$ (read "not P) indicates its negation.

Definition A **TRUTH TABLE** is a table that provides the truth values of a statement by considering all possible true/false combinations of the statement's components.

P	$\sim P$
T	F
F	T

→ If P is true, then $\sim P$ is false.
→ When P is false, $\sim P$ is true.

Exercise #2 Give the negation of each statement.

a) Christopher Columbus crossed the Atlantic Ocean.
Christopher Columbus did not cross the Atlantic Ocean

b) $2+5=7$ $2+5 \neq 7$

c) Her aunt's name is Lucia. Her aunt's name is not Lucia

d) $y>12$ $y \leq 12$

e) $q \geq 5$ $q < 5$

Note: **QUANTIFIERS** are used in extensively in mathematics to indicate how many cases of a particular situation exist.

UNIVERSAL QUANTIFIERS:	EXISTENTAIL QUANTIFIERS:
all, each, every, no(ne)	*some, there exists, at least one*

Exercise #3 Give the negation of each statement.

a) Some cats have fleas. No cat has fleas.

b) Some cats do not have fleas. All cats have fleas.

c) No cats have fleas. Some cats have fleas.

d) All jokes are funny. Some jokes are not funny

e) Every dog has its day. At least one dog does not have it's day

f) No computer repairman can play blackjack. At least one computer repairman can play blackjack.

COMPOUND STATEMENTS

Statements can be combined to form compound statements using **logical connectives** (connectives) such as *and, or, not, and if…then.*

Exercise #4 Decide whether each of the following statements is compound.

a) My brother got married in London. NO

b) I read the Chicago Tribune and I read the New York Times. YES

c) If Julie sells her quota, then Bill will be happy. YES

Connective	Symbol	Type of Statement
and	$\wedge$	Conjunction
or	$\vee$	Disjunction
not	$\sim$	Negation

Exercise #5 Let p represent the statement "She has green eyes" and let q represent the statement "He is 48 years old". Translate each symbolic compound statement into words.

a) $\sim p$ She doesn't have green eyes.

b) $p \wedge q$ She has green eyes and he is 48 years old.

c) $\sim(\sim p \wedge q)$ It is not the case that she doesn't have green eyes and he is 48 years old.

Exercise #6 Let p represent the statement "Chris collects videotapes" and let q represent the statement "Jack plays the tuba." Convert each of the following compound statement into symbols.

a) Chris collects videotapes and Jack does not play the tuba. $p \wedge \sim q$

b) Chris does not collect videotapes or Jack plays the tuba. $\sim p \vee q$

c) Neither Chris collects videotapes nor Jack plays the tuba. $\sim p \wedge \sim q$ OR $\sim(p \vee q)$

Definition A **CONJUNCTION** is a statement of the form ***P* and *Q*.** $\boxed{P \wedge Q}$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

For the conjunction to be true, it is necessary for P to be true *and* Q to be true.

Exercise #7 Let P="Babe Ruth played baseball" and Q="4 + 3 <5." Classify as true or false:

T F

a) $P \wedge Q$ $T \wedge F$ False

b) $P \wedge \sim Q$ $T \wedge T$ True

Definition A **DISJUNCTION** is a statement of the form ***P* or *Q*.**

$\boxed{P \vee Q}$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

A disjunction is false only if P and Q are both false.

Example: You can join the Math Club if you have an A average or you are enrolled in a mathematics class.

Exercise #8 Let P="Babe Ruth played baseball" (T) and Q="4 + 3 <5." (F) Classify as true or false:

a) $P \vee Q$ T V F TRUE

b) $P \vee \sim Q$ T V T TRUE.

Exercise #9 Statement P is true, Q is true, and R is false. Classify each statement as true or false.

a) $P \wedge Q$
T ∧ T
T

b) $Q \wedge R$
T ∧ F
F

c) $P \wedge (Q \vee R)$
T ∧ (T V F)
T ∧ T
T

Exercise #10 Construct a truth table for each compound statement.

a) $\sim p \wedge q$

p	q	~p	~p ∧ q
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

b) $(q \vee \sim p) \vee \sim q$

p	q	~p	q V ~p	~q	(q V ~p) V ~q
T	T	F	T	F	T
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	T	T	T

Definition An **IMPLICATION or CONDITIONAL** is a statement of the form **"If *P*, then *Q*."**

Note: *P* is called the *antecedent (*or *hypothesis)*
Q is called the *consequent (*or *conclusion)*

$$\boxed{P \rightarrow Q}$$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The conditional statement makes a promise and fails to satisfy the conditions of this promise only when P is true and Q is false.

Example Consider the claim, "If you are good, then I'll give you a dollar."
The only way the claim is false is when "you are good, but I don't give you the dollar."

Consider the statement made by a politician, Senator Bridget Terry, "if I am elected, then taxes will go down."
The only way the claim is false is when she is elected, but the taxes do not go down.

Definition Two statements are **logically equivalent** if their truth values are the same for all possible true/false combinations of their components.

Definition A TAUTOLOGIE is a statement that is true for all possible truth value of its components.

Exercise #11 Form a truth table and determine all possible truth values for the given statement. Is the given statement a tautology?

$$[(P \rightarrow Q) \wedge P] \rightarrow Q$$

P	Q	$P \rightarrow Q$	$(P \rightarrow Q) \wedge P$	$[(P \rightarrow Q) \wedge P] \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

yes, this statement is a tautology.

DEMORGAN'S LAWS

In the study of logic, DeMorgan's Laws (19th century) are used to describe the negation of the conjunction ($\wedge$) and disjunction ($\vee$).

1. $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ The negation of a conjunction is the disjunction of negations.
2. $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$ The negation of a disjunction is the conjunction of negations.

Proof of DeMorgan's first law

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

the same, therefore the statements are equivalent

Proof of DeMorgan's second law

P	Q	$\sim($	$P \vee Q)$	$\sim P$	$\wedge$	$\sim Q$
T	T	F	T	F	F	F
T	F	F	T	F	F	T
F	T	F	T	T	F	F
F	F	T	F	T	T	T

the same, therefore the statements are equivalent

Exercise #12 Use DeMorgan's Laws to write the negation of the given statement.

a) $P \wedge Q$

$\sim(P \wedge Q) \equiv \sim P \vee \sim Q$

b) Mary is an accountant or hamburgers are health food.

Mary is not an accountand and hamburgers are not healthy food.

c) It is cold and snowing.

It's not cold OR it's not snowing

Exercise #13 Use a truth table to show that $[P \wedge \sim Q]$ is the negation of $P \to Q$.

$$\sim(P \to Q) \equiv P \wedge \sim Q$$

P	Q	$\sim$	$(P \to Q)$	P	$\wedge$	$\sim Q$
T	T	F	T	T	F	F
T	F	T	F	T	T	T
F	T	F	T	F	F	F
F	F	F	T	F	F	T

the same, therefore the statements are equivalent

Exercise #14 Write the negation of the given statement.

$P \to Q$

a) If it is medicine, then it tastes bad.
P Q

It is medicine and it doesn't taste bad

$P \to Q$

b) If I am good, then I can go to the movie.
P Q

I am good and I can't go to the movie

$P \vee Q$

c) You can pay me now or you can pay me later.
P Q

You can't pay me now and you can't pay me later.

$P \wedge Q$

d) It is summer and there is no snow.
P Q

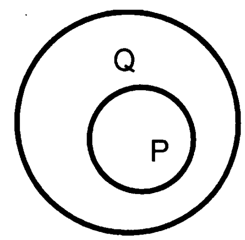
It is not summer or there is snow.

CONVERSE . INVERSE. CONTRAPOSITIVE

Lewis Carroll, the author of *Alice's Adventures in Wonderland* and *Through the looking Glass*, was a mathematician teacher who wrote stories as a hobby. His books contain many amusing examples of both good and deliberately poor logic. Consider the following conversation held at the Mad Hatter's tea Party.

" Then you should say what you mean,", the March Hare went on.
"I do,", Alice hastily replied; "at least – at least I mean what I say – that's the same thing, you know."
" Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
"You might just as well say, "added the March Hare, "that 'I like what I get' is the same as 'I get what I like'!"
"You might just as well say," added the Dormouse, who seemed to be talking in his sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"
"It *is* the same thing with you," said the Hatter, and here the conversation dropped, and the party went silent for a minute.

Carroll is playing here with pairs of related statements and the Hatter, the Hare, and the Dormouse are right: the sentences in each pair do not say the same thing at all.



Conditional statement: $\boxed{P \to Q}$ If P, then Q. (P implies Q)

Its **CONVERSE** $\boxed{Q \to P}$ If Q, then P.

- The converse of a conditional statement is formed by interchanging its hypothesis and conclusion.
- The converse of a true statement may be false. It is also possible that it may be true, but in either case a statement and its converse do not have the same meaning.

Its **INVERSE**: $\boxed{\sim P \to \sim Q}$ If *not P*, then *not Q*.

- The inverse of a conditional statement is formed by denying both its hypothesis and conclusion.

Its **CONTRAPOSITIVE**: $\boxed{\sim Q \to \sim P}$ If *not Q*, then *not P*.

- The contrapositive of a conditional statement is formed by interchanging its hypothesis and conclusion and denying both.

Exercise #15 Write each statement in the form "if p, then q."

a) You'll be sorry if I go. If I go, then you'll be sorry.

b) Today is Friday only if yesterday was Thursday.
If today is Friday, then yesterday was Thursday

c) All nurses wear white shoes. If you are a nurse, then you wear white shoes.

Exercise #16 State the hypothesis and the conclusion of each statement.

a) If you go to the game, then you will have a great time.

Hypothesis: if you go to the game

Conclusion: you will have a great time

b) If two cords of a circle have equal lengths, then the arcs of the chords are congruent.

Hypothesis: two cords of a circle have equal lenghts

Conclusion: the arcs of the chords are congruent

b) Vertical angles are congruent when two lines intersect.

Hypothesis: if two lines intersect

Conclusion: vertical angles are congruent

Exercise #17 Identify the relationship of each of the lettered statements to the given statement if possible. Write "converse," "inverse," "contrapositive," "original statement," or "none,", as appropriate.

$P \rightarrow Q$

converse: $Q \rightarrow P$

inverse: $\sim P \rightarrow \sim Q$

contrapositive: $\sim Q \rightarrow \sim P$

if a kangaroo is a lady, then it doesn't need a handbag

Lady kangaroos do not need handbags.

a) If a kangaroo is not a lady, it needs a handbag. inverse

b) If it needs a handbag, then it is not a lady kangaroo. contrapositive

c) A kangaroo does not need a handbag if it is a lady. original statement

Exercise #18 Write the inverse, converse, and contrapositive of the following statement:

$P \longrightarrow Q$

"If you live in Atlantis, then you need a snorkel."

$\sim P \rightarrow \sim Q$ a) Inverse: If you don't live in Atlantis, then you don't need a snorkel

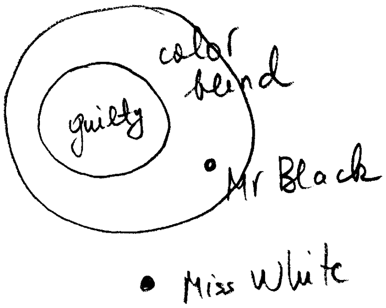
$Q \rightarrow P$ b) Converse: If you need a snorkel, then you live in Atlantis

$\sim Q \rightarrow \sim P$ c) Contrapositive: if you don't need a snorkel, then you don't live in Atlantis

VALID ARGUMENTS

Suppose that during a trial a lawyer claims that, from the evidence presented, the guilty person is obviously color-blind and that everyone on the jury accepts this as true. Then he produces proof that Mr. Black is color-blind. Must the jury conclude that Mr. Black is guilty? Suppose also that it is established that Miss White is not color-blind. Must Miss White be innocent?

If a person is guilty, then he is color blind.



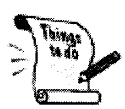
* We can't tell where Mr. Black belongs, so no conclusion is justified.

* Miss White is outside the larger circle, she cannot be inside the smaller one, and so she is not the guilty person.

Definition An **ARGUMENT** is a set of statements called **premises**, followed by a statement called the **conclusion.** In a **VALID ARGUMENT**, the truth of the premises forces a conclusion that must also be true.

LAW OF DETACHMENT

1. $P \to Q$	Premise 1
2. P	Premise 2
C. Q	Conclusion



Give the symbolic form and prove the Law of Detachment

$[(P \to Q) \wedge P] \to Q$ we'll show it's a tautologic

P	Q	$(P \to Q)$	$\wedge$	P	$[(P \to Q) \wedge P] \to Q$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	F	F	T
F	F	T	F	F	T

ATTENTION!!! INVALID ARGUMENT

1. $P \to Q$	Premise 1
2. Q	Premise 2
C. P	Conclusion

Exercise #19 Use the Law of Detachment to draw a conclusion.

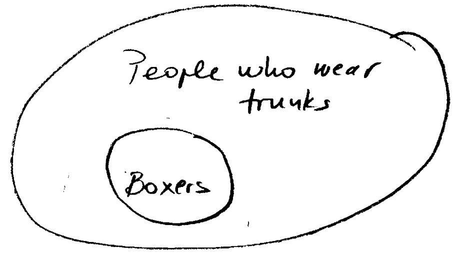
a) If two angles are complementary, the sum of their measures is 90^o. $\angle 1$ and $\angle 2$ are complementary.

CONCLUSION: Their sum is 90^o

e) If it gets hot this morning, we will have to turn on the air conditioner. It is hot this morning.

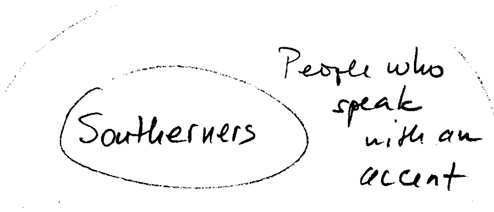
CONCLUSION: We'll have to turn on the air conditioner

Exercise #20 Decide whether each argument is valid or invalid.



a) All boxers wear trunks.
<u>Chris Mader is a boxer.</u>
Chris Mader wears trunks.

Valid



b) All Southerners speak with an accent.
<u>Bill speaks with an accent.</u>
Bill is a Southerner.

invalid

LAW OF NEGATIVE INFERENCE

1. $P \to Q$	Premise 1
2. $\sim Q$	Premise 2
C. $\sim P$	Conclusion

<u>Give the symbolic form and prove the Law of Negative Inference</u>

$[(P \to Q) \wedge \sim Q] \to \sim P$ and we'll show it's a tautologie

P	Q	$(P \to Q)$	$\wedge$	$\sim Q$	$[(P \to Q) \wedge \sim Q]$	$\to$	$\sim P$
T	T	T	F	F	F	T	F
T	F	F	F	T	F	T	F
F	T	T	F	F	F	T	T
F	F	T	T	T	T	T	T

tautologie

Exercise #21 Use the Law of Negative Inference to draw a conclusion.

a) If Tom doesn't finish the job then I will not pay him. I did pay Tom for the job.

CONCLUSION: Tom finished the job.

b) If the traffic light changes, then you can travel through the intersection. You cannot travel through the intersection.

CONCLUSION: The light did not change.

LAW OF SYLLOGISM

1. $P \rightarrow Q$	Premise 1
2. $Q \rightarrow R$	Premise 2
C. $P \rightarrow R$	Conclusion

Give the symbolic form and prove the Law of Syllogism

$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$

P	Q	R	$(P \rightarrow Q)$	$\wedge$	$(Q \rightarrow R)$	$\rightarrow$	$(P \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	T
T	F	F	F	F	T	T	F
F	T	T					
F	T	F					
F	F	T					
F	F	F					

continue the other half.

Exercise #22 Use the Law of Syllogism to draw a conclusion.

If Izzi lives in Chicago, then she lives in Illionois. If a person lives in Illinois, then she lives in the Midwest.

CONCLUSION: If Izzi lives in Chicago, then she lives in the Midwest

Exercise #23 | Determine which arguments are valid.

a) 1. If Bill and Mary stop to visit, I'll prepare a meal.
   2. Bill stopped to visit at 5 p.m.

   C. I prepared a meal.

   VALID: YES **NO**

b) 1. If it turns cold and snows, I'll build a fire in the fireplace.
   2. The temperature began to fall around 3 p.m.
   3. It began snowing before 5 p.m.

   C. I built a fire in the fireplace.

   VALID: **YES** NO

LAW OF DENIAL

1. $P \vee Q$		Premise 1
2. $\sim Q$		Premise 2
C.	P	Conclusion

Give the symbolic form and prove the Law of Denial

$[(P \vee Q) \wedge \sim Q] \rightarrow P$ is a tautology

P	Q	$[(P \vee Q)$	$\wedge$	$\sim Q]$	$\rightarrow$	P
T	T	T	F	F	T	T
T	F	T	T	T	T	T
F	T	T	F	F	T	F
F	F	F	F	T	T	F

Exercise #24 | Use the Law of Denial to draw a conclusion.

Terry is sick or hurt.
Terry is not hurt.

CONCLUSION: Terry is sick.

References
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Harold R. Jacobs, Geometry, W.H. Freeman and Company, 1974