10.8 & 10.9
Taylor and Maclaurin Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems:

- Which functions have power series representations?
- How can we find such representations?

We start by **supposing that *f* is any function that can be represented by a power series.**

(1) $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, for any x such that $|x-a| < R$

Let's find the coefficients c_n .

Therefore,

<u>Theorem</u>

> If f has a power series representation (expansion) at a, that is, if
>
> $$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \ |x-a| < R$$
>
> then its coefficients are given by the formula $c_n = \dfrac{f^{(n)}(a)}{n!}$

We see that *if* f has a power series expansion at a, then it must be of the following form:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$$

the **Taylor series of the function *f* at *a*.**

For the special case $a=0$ the Taylor series becomes

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$

the **Maclaurin series of the function *f* at *a*.**

<u>Note</u> We have shown that *if* f can be represented as a power series about a, then f is equal to the sum of its Taylor series. But there are functions that are not equal to the sum of their Taylor series.

Exercise 1 Find the Maclaurin series of the function $f(x)=e^x$ and its radius of convergence.

Conclusion: *If* e^x has a power series expansion at 0, then $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

<u>Question:</u> Under what circumstances is a function equal to the sum of its Taylor series?
In other words, if f has derivatives of all orders, when is it true that

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \ ?$$

<u>Theorem</u>

If $f(x)=T_n(x)+R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty}R_n(x)=0 \quad \text{for } |x-a|<R,$$

then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n, \quad \text{with } |x-a|<R$$

How to Estimate $R_n(x)$

<u>Theorem – The Remainder Estimation Theorem</u>

If $\left|f^{(n+1)}(x)\right|\le M$ for $|x-a|\le d$ (for some interval around a), then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$\left|R_n(x)\right|\le\frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a|\le d$$

Exercise 2 Prove that e^x is equal to the sum of its Maclaurin series.

Exercise 3 Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Note: We have two power series for e^x, the Maclaurin series in Exercise 2 and the Taylor series in Exercise 3. The first is better if we are interested in values of x near 0 and the second is better if x is near 2.

Exercise 4 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

Exercise 5 Find the Maclaurin series for $\cos x$.

Exercise 6 Find the Maclaurin series for $f(x) = x\cos x$.

Exercise 7 Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\frac{p}{3}$.

Some important Maclaurin series that we have derived in this section and the preceding one

$\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$	for any $x \in (-1,1)$
$e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	for any $x \in (-\infty,\infty)$
$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	for any $x \in (-\infty,\infty)$
$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	for any $x \in (-\infty,\infty)$
$\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	for any $x \in [-1,1]$

Exercise 8 a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

b) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Exercise 9 Find the first three nonzero terms in the Maclaurin series for

a) $e^x \sin x$

b) $\tan x$.