

10.8 & 10.9

Taylor and Maclaurin Series

In the preceding section we were able to find power series representations for a certain restricted class of functions. Here we investigate more general problems:

- Which functions have power series representations?
- How can we find such representations?

We start by **supposing that f is any function that can be represented by a power series.**

$$(1) f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \text{ for any } x \text{ such that } |x-a| < R$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

Let's find the coefficients c_n .

$$\text{let } x=a \Rightarrow \boxed{f(a) = c_0}$$

Differentiate the series in Eq. (1) term by term:

$$(2) f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \quad |x-a| < R$$

$$\text{let } x=a \Rightarrow \boxed{f'(a) = c_1}$$

Differentiate both sides of Eq. (2):

$$(3) f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \dots \quad |x-a| < R$$

$$\text{let } x=a \Rightarrow f''(a) = 2c_2$$

$$\boxed{\frac{f''(a)}{2} = c_2}$$

Differentiate both sides of Eq. (3):

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \dots \quad |x-a| < R$$

$$\text{let } x=a \Rightarrow \boxed{f'''(a) = 2 \cdot 3c_3 = 3!c_3} \quad \left(\frac{f'''(a)}{3!} = c_3 \right)$$

$$\text{so } \boxed{c_n = \frac{f^{(n)}(a)}{n!}, \forall n \geq 0}$$

Therefore,

Theorem

If f has a power series representation (expansion) at a , that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R$$

then its coefficients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$

We see that if f has a power series expansion at a , then it must be of the following form:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots \end{aligned}$$

the Taylor series of the function f at a .

For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x)^n = f(0) + \frac{f'(0)}{1!} (x) + \frac{f''(0)}{2!} (x)^2 + \frac{f'''(0)}{3!} (x)^3 + \dots$$

the Maclaurin series of the function f at a .

Note

We have shown that if f can be represented as a power series about a , then f is equal to the sum of its Taylor series. But there are functions that are not equal to the sum of their Taylor series.

Exercise 1 Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

$$\begin{aligned} f(x) &= e^x, \quad f(0) = 1 \\ f^{(n)}(x) &= e^x, \quad \forall n \\ f^{(n)}(0) &= 1 \\ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= \left| 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right| = \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

MacLaurin series of $f(x) = e^x$

Find x for which the series converges.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = |x| \cdot 0 = 0, \quad \forall x$$

$\therefore \sum \left| \frac{1}{n!} x^n \right|$ is (C) $\Rightarrow \sum \frac{x^n}{n!}$ is (C) $\forall x \rightarrow R = \infty$

Conclusion: If e^x has a power series expansion at 0, then $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Question:

Under what circumstances is a function equal to the sum of its Taylor series?
In other words, if f has derivatives of all orders, when is it true that

$$\text{(*) } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n ?$$

by def, (*) means that $f(x)$ is the limit of the sequence of partial sums

$$f(x) = \lim_{n \rightarrow \infty} T_n(x) \quad | \quad T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$n^{\text{th-degree Taylor polynomial of } f \text{ at } a}$

let $(R_n(x) = f(x) - T_n(x))$ remainder of the Taylor series

$$f(x) = T_n(x) + R_n(x)$$

if $\lim_{n \rightarrow \infty} R_n(x) = 0$, then $f(x) = \lim_{n \rightarrow \infty} T_n(x)$

$$\text{so } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Theorem

If $f(x) = T_n(x) + R_n(x)$, where T_n is the $n^{\text{th-degree Taylor polynomial of } f \text{ at } a}$ and

$$\lim_{n \rightarrow \infty} R_n(x) = 0 \text{ for } |x-a| < R,$$

then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \text{ with } |x-a| < R$$

How to Estimate $R_n(x)$ Theorem – The Remainder Estimation Theorem

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ (for some interval around a), then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \text{ for } |x-a| \leq d$$

FACT:

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \forall x \in \mathbb{R} \quad | \quad \begin{array}{l} \text{know } \sum \frac{x^n}{n!} (C) \neq x \\ \text{ex. } \Rightarrow \lim_{n \rightarrow \infty} a_n = 0 \end{array}$$

Exercise 2 Prove that e^x is equal to the sum of its Maclaurin series.

$f(x) = e^x$, $f^{(n+1)}(x) = e^x$, $\forall n$
 if d is any positive # and $|x| \leq d$
 $|f^{(n+1)}(x)| = e^x \leq e^d$

Let $M = e^d$, $a = 0$

$$|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1} \quad \text{for } |x| \leq d$$

$$\text{but } \lim_{n \rightarrow \infty} \frac{e^d}{(n+1)!} |x|^{n+1} = e^d \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

By squeeze th $\Rightarrow \lim_{n \rightarrow \infty} |R_n(x)| = 0, \forall x$

$$\therefore \lim_{n \rightarrow \infty} R_n(x) = 0, \forall x$$

$$\therefore \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x}$$

Exercise 3 Find the Taylor series for $f(x) = e^x$ at $a = 2$.

$f^{(n)}(2) = e^2$, $a = 2$ via the def. of a Taylor series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$$

it can be shown $R = \infty$ (like in EX.1)
 $\lim_{n \rightarrow \infty} R_n(x) = 0$ (like in EX.2)

$$\therefore \boxed{e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n \text{ for all } x}$$

Note: We have two power series for e^x , the Maclaurin series in Exercise 2 and the Taylor series in Exercise 3. The first is better if we are interested in values of x near 0 and the second is better if x is near 2.

Exercise 4 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x .

Exercise 5 Find the Maclaurin series for $\cos x$.

Exercise 6 Find the Maclaurin series for $f(x) = x \cos x$.

Exercise 7 Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\frac{\pi}{3}$.

$$\textcircled{4} \quad f(x) = \sin x \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

MacLaurin series

$$\begin{array}{ll}
 f(x) = \sin x & f(0) = 0 \\
 f'(x) = \cos x & f'(0) = 1 \\
 f''(x) = -\sin x & f''(0) = 0 \\
 f'''(x) = -\cos x & f'''(0) = -1 \\
 f^{(iv)}(x) = \sin x & f^{(iv)}(0) = 0
 \end{array}$$

and the pattern continues

$$\begin{aligned}
 \text{MacLaurin series is: } & f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \\
 & + \frac{f^{(iv)}(0)}{4!} x^4 + \frac{f^{(v)}(0)}{5!} x^5 + \dots = \\
 = & \left| x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} (-1)^n \right|
 \end{aligned}$$

$$|f^{(n+1)}(x)| \leq 1 \quad \forall x$$

$$\text{let } M = 1$$

$$|R_n(x)| \leq \frac{1}{(n+1)!} |x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$$

$$\text{but } \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

$$\begin{aligned}
 \text{By Squeeze Th} \Rightarrow & \lim |R_n(x)| = 0 \\
 \Rightarrow & \lim R_n(x) = 0
 \end{aligned}$$

$$\text{So } \left| \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \forall x \right|$$

(5) We could proceed directly as in (4)
but it's easier to differentiate the
Maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \quad \forall x$$

$$\frac{d}{dx}(\sin x) = \frac{d}{dx} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$$

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \forall x \end{aligned}$$

(6) $f(x) = x \cos x$

$$x \cos x = x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)$$

$$= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}$$

(7) $f(x) = \sin x$
 Taylor series of f at $x = \frac{\pi}{3}$ 15

$$\sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{3}\right)}{n!} \left(x - \frac{\pi}{3}\right)^n = f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!} \left(x - \frac{\pi}{3}\right) + \\ + \frac{f''\left(\frac{\pi}{3}\right)}{2!} \left(x - \frac{\pi}{3}\right)^2 + \dots$$

$$f(x) = \sin x \quad f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$f'(x) = \cos x \quad f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f''(x) = -\sin x \quad f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = -\cos x \quad f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

and this pattern repeats indefinitely

$$\left| \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!} \left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!} \left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!} \left(x - \frac{\pi}{3}\right)^3 + \dots \right.$$

$$|f^{(n+1)}(x)| \leq 1 \quad \forall x$$

$$\text{let } M = 1$$

$$|R_n(x)| \leq \frac{1}{(n+1)!} \left|x - \frac{\pi}{3}\right|^{n+1}$$

but $\lim_{n \rightarrow \infty} \frac{\left|x - \frac{\pi}{3}\right|^{n+1}}{(n+1)!} = 0$

$$\text{so } \lim |R_n(x)| = 0$$

$$\lim R_n(x) = 0$$

$$\left| \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!} \left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!} \left(x - \frac{\pi}{3}\right)^{2n+1} \right|$$

$$\textcircled{2} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots, \neq x$$

$$\textcircled{3} \quad x \mapsto -x^2 \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

integrate term by term:

$$\begin{aligned} \int e^{-x^2} dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \right) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot n!} + \dots \end{aligned}$$

The series converges for all x because

the original series for e^{-x^2} converges for all x

$$\textcircled{3} \quad \int_0^1 e^{-x^2} dx = \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \dots \right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{11520} = 1320$$

$$\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475$$

The Rel. Series Estimation Thm says.

error is less than $\frac{1}{115!} = \frac{1}{1320} < 0.001$

~~(c) by~~ ~~term~~ ~~term~~ ~~term~~ ~~term~~
~~term~~ ~~term~~ ~~term~~ ~~term~~ ~~term~~