9.1 Solutions, Slope Fields, and Euler's Method

In-class work:

Model of Population Growth

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).

Identify and name the variables in this model:

Write an equation that models the problem (Equation (1)):

Having formulated a model, let's look at its consequences.

Think of a solution of Equation (1):

Take a look at the family of solutions:

Note: Equation (1) is appropriate for modeling population growth under ideal conditions, but a more realistic model must reflect the fact that a given environment has limited resources. Many populations start by increasing in an exponential manner, but the population levels off when it approaches its carrying capacity.

General Differential Equations

Definitions

1) A **differential equation** is an equation that contains an unknown function and one or more of its derivatives.

2) The **order** of a differential equation is the order of the highest derivative that occurs in the equation.

Examples

a) $y' = xy$ (It is understood that y is an unknown function of x)

b) In general, $\frac{dy}{dx} = f(x, y)$ is a first order differential equation

3) A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

4) **To solve** a differential equation means to find all possible solutions of the equation.

Example Solve $y' = x^3$.

Note: In general, solving a differential equation is not an easy matter. There is no systematic technique that enables us to solve all differential equations. We will see how to **draw rough graphs of solutions** even when we have no explicit formula. We will also learn how to **find numerical approximations to solutions**.

1. Show that every member of the family of functions $y = \frac{1+ce^t}{1-ce^t}$ is a solution of the differential equation $y' = \frac{1}{2}\left(y^2 - 1\right)$.

Note: We are usually not as interested in finding a family of solutions (the general solution) as we are in finding a solution that satisfies some additional requirements. In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Geometrically, when we impose an initial condition, we look at the family of solution curves and pick the one that passes through the point (t_0, y_0). Physically, this corresponds to measuring the state of a system at time t_0 and using the solution of the initial-value problem to predict future behavior of the system.

2. Find a solution of the differential equation $y' = \frac{1}{2}\left(y^2 - 1\right)$ that satisfies the initial condition $y(0) = 2$.

Slope Fields (Direction Fields) and Euler's Method

It is impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. Despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's method).

Slope Fields (Direction Fields)

3. Sketch the graph of the solution of the initial-value problem $y' = x + y$, $y(0) = 1$.

Solution:

a) We start by computing the slope at several points :

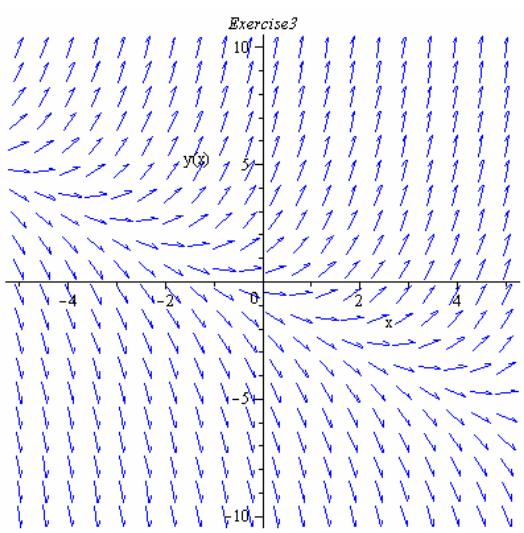
x	-2	-1	0	1	2	-2	-1	0	1	2
y	0	0	0	0	0	1	1	1	1	1
$y' = x + y$										

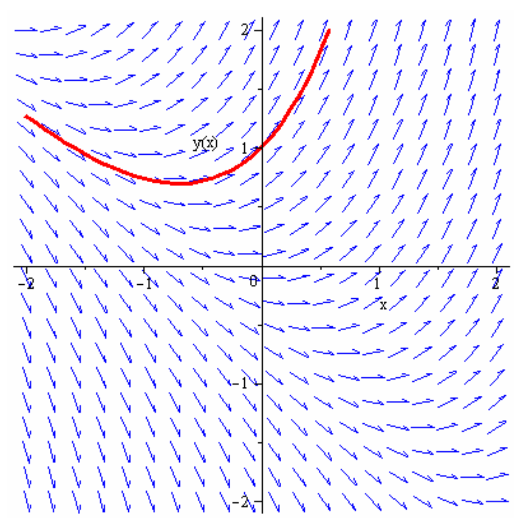
b) We draw short line segments with these slopes at these points. The result is the direction field shown in the first figure. The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

c) To draw the solution curve through (0,1) we start at (0,1) and move to the right in the direction of the line segment (which has slope 1). We continue to draw the solution curve so that it moves parallel to the nearby line segments. Returning at (0,1), we draw the solution curve to the left as well. The resulting solution curve is shown in the second figure.

<u>Maple commands for the first graph</u> (it plots a slope field for $y' = x + y$)
with(DEtools):
dfieldplot(diff(y(x),x)=x+y(x),y(x),x=-2..2,y=-2..2,color=blue,title='Exercises3');

<u>Maple commands for the second graph</u> (it adds an initial condition and plots a curve through it)
DEplot(diff(y(x),x)=x+y(x),y(x),x=-2..2,[[y(0)=1]],y=-2..2);





4. a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.
b) Use part (a) to sketch the solution curve that passes through the origin.

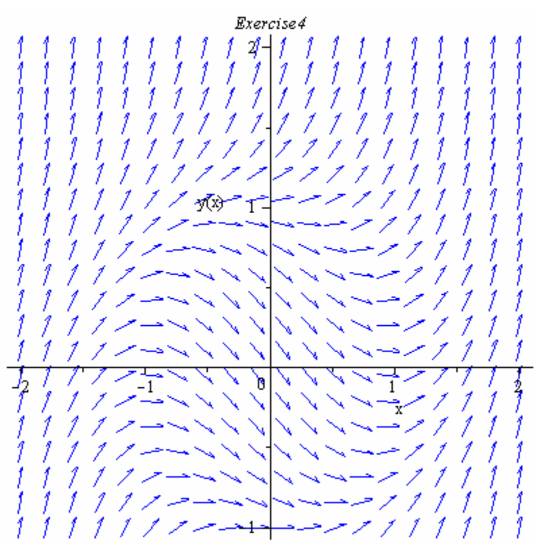
Solution:

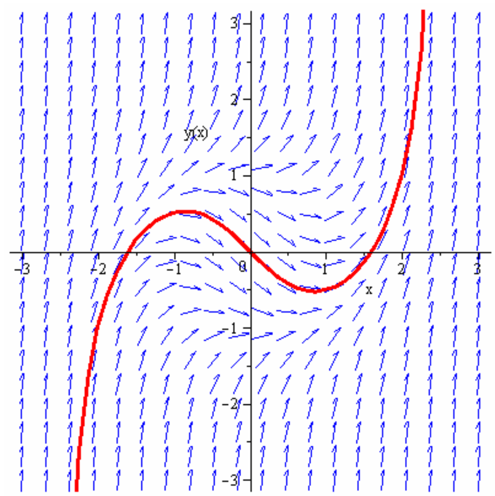
We start by computing the slope at several points :

x	-2	-1	0	1	2	-2	-1	0	1	2
y	0	0	0	0	0	1	1	1	1	1
$y' = x^2 + y^2 - 1$										

We draw short line segments with these slopes at these points. The result is the direction field shown in the first figure. The direction field allows us to visualize the general shape of the solution curves by indicating the direction in which the curves proceed at each point.

To draw the solution curve through (0,0) we start at (0,0) and move to the right in the direction of the line segment (which has slope -1). We continue to draw the solution curve so that it moves parallel to the nearby line segments. Returning at (0,0), we draw the solution curve to the left as well. The resulting solution curve is shown in the second figure.





5. (Exercise 7/9.1) Write an equivalent first-order differential equation and initial condition for y.

$$y = -1 + \int_1^x \left(t - y(t)\right) dt$$

Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations. In general, Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short time, look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field. Euler's method does not produce the exact solution to an initial-value problem – it gives approximations. But by decreasing the step size (and therefore increasing the number of midcourse corrections), we obtain successively better approximations to the exact solution.

We will illustrate the method on the initial-value problem in the next exercise.

6. Use Euler's method with step size 0.1 to calculate the first three approximations to the given initial value problem for the specified increment size.

$$y' = x + y, \quad y(0) = 1, \; h = 0.1 \; (dx = 0.1).$$