6.6 Moments and Centers of Mass

Our main objective here is to find the point *P* on which a thin plate of any given shape balances horizontally.
This point is called the **center of mass** (or center of gravity) of the plate.

1. We first consider two masses m_1 and m_2 that are attached to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum.
By the *Law of the Lever* (an experimental fact discovered by Archimedes),
the rod will balance if

$$(1) \qquad m_1 d_1 = m_2 d_2 \,.$$

(Think of a lighter person balancing a heavier one on a seesaw by sitting farther away from the center.)

2. Suppose the rod lies along the *x*-axis with m_1 at x_1 and m_2 at x_2 and the center of mass at $\overline{x}$.
Equation (1) gives

$$m_1\left(\overline{x} - x_1\right) = m_2\left(x_2 - \overline{x}\right)$$
$$m_1\overline{x} + m_2\overline{x} = m_1 x_1 + m_2 x_2$$

$$(2) \qquad \overline{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

The numbers $m_1 x_1$ and $m_2 x_2$ are called the **moments of masses** m_1 **and** m_2 **(**with respect to the origin), $m_1 x_1 + m_2 x_2$ is called the **moment of the system**(with respect to the origin), and $m_1 + m_2$ is the total mass.

3. Masses Along a Line

In general, if we have a system of *n* particles with masses $m_i, i = \overline{1,n}$, located at the points $x_i, i = \overline{1,n}$ on the *x*-axis, it can be shown similarly that the center of mass of the system is located at

$$(3) \qquad \overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

The number $\sum_{i=1}^{n} m_i x_i$ is called the **moment of the system** with respect to the origin, and $\sum_{i=1}^{n} m_i = m$ is the **total mass** of the system.
The center of mass is that point where a single particle of mass *m* would have the same moment as the system.

4. **Masses Distributed over a Plane Region**

Now we consider a system of particles with masses $m_i, i = \overline{1,n}$ located at the points $(x_i, y_i), i = \overline{1,n}$ in the *xy*-plane. By analogy with the one-dimensional case, we define
the **moment of the system about the *y*-axis** to be

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the **moment of the system about the *x*-axis** to be

$$M_x = \sum_{i=1}^{n} m_i y_i \ .$$

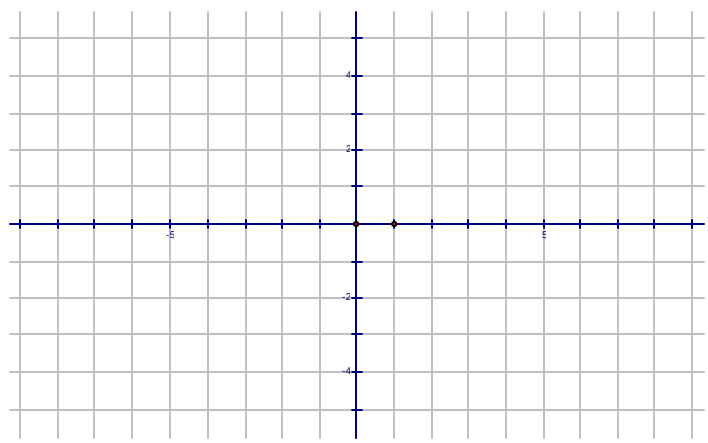
M_y measures the tendency of the system to rotate about the *y*-axis and M_x measures the tendency to rotate about the *x*-axis.

As in the one-dimensional case, the coordinates $(\overline{x}, \overline{y})$ of the center of mass are given in terms of the moments by the formulas

$$(4) \qquad \overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M_y}{m} \qquad\qquad \overline{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{M_x}{m} \ .$$

The center of mass is that point where a single particle of mass *m* would have the same moments as the system.

<u>Exercise 1</u> Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1,1), (2,-1)$, and $(3,2)$.



5. **Plates Bounded by One Curve**

Consider a flat plate with uniform density d that occupies a region *R* in the plane. We wish to locate the center of mass of the plate, which is called the centroid of *R* (when the density is constant).

We'll use the following physical principle:

The symmetry principle

If a region *R* is symmetric about a line *l*, then the center of mass of *R* lies on *l*.

Therefore, <u>the center of mass of a rectangle is its center.</u>

Suppose the region *R* lies between the lines $x = a$ and $x = b$, above the *x*-axis , and beneath the graph of *f*, where *f* is a continuous function.

- Divide the interval $[a,b]$ into *n* subintervals $[x_{i-1}, x_i]$, $i = \overline{1,n}$, of equal length Δx
- Choose the sample points $\overline{x_i}$, the midpoint of the ith subinterval, $i = \overline{1,n}$ and construct the rectangular approximation of the region.
- The center of mass of the ith approximating rectangle R_i is its center $\left(\overline{x_i}, \frac{1}{2} f\left(\overline{x_i}\right)\right)$.
- The mass of the rectangle R_i is

$$d\, f\left(\overline{x_i}\right)\Delta x \quad \text{(mass=density times area).}$$

- Find the moment of the rectangle R_i about the *y*-axis,

$$M_y\left(R_i\right) = \left[d\, f\left(\overline{x_i}\right)\Delta x\right]\overline{x_i}$$

- Adding these moments, we obtain the moment of the polygonal approximation to *R*, then by taking the limit as $n \to \infty$ we obtain the moment of *R* about the *y*-axis:

$$\boxed{M_y = \lim_{n\to\infty}\sum_{i=1}^{n} d\,\overline{x_i}\, f\left(\overline{x_i}\right)\Delta x = \int_a^b d\, x f\left(x\right) dx}$$

- Find the moment of the rectangle R_i about the *x*-axis,

$$M_x\left(R_i\right) = \left[d\, f\left(\overline{x_i}\right)\Delta x\right]\frac{1}{2} f\left(\overline{x_i}\right) = d \cdot \frac{1}{2}\left[f\left(\overline{x_i}\right)\right]^2 \Delta x$$

- Adding these moments, we obtain the moment of the polygonal approximation to *R*, then by taking the limit as $n \to \infty$ we obtain the moment of *R* about the *x*-axis:

$$\boxed{M_x = \lim_{n\to\infty}\sum_{i=1}^{n} d \cdot \frac{1}{2}\left[f\left(\overline{x_i}\right)\right]^2 \Delta x = \int_a^b d\, \frac{1}{2}\left[f\left(x\right)\right]^2 dx}$$

- The mass of the plate is the product of its density and its area:

$$m = d\,A = \int_a^b d\,f(x)\,dx$$

- The center of mass of R is given by the coordinates:

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b d\,xf(x)\,dx}{m};$$

if the density is constant, then $\bar{x} = \dfrac{d\int_a^b xf(x)\,dx}{d\int_a^b f(x)\,dx} = \dfrac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx} = \dfrac{\int_a^b xf(x)\,dx}{A}$

$$\bar{y} = \frac{M_x}{m} = \frac{\int_a^b d\,\frac{1}{2}\left[f(x)\right]^2 dx}{m};$$

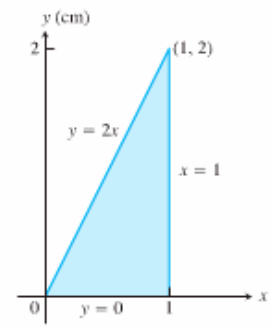
if the density if constant, then

$$\bar{y} = \frac{d\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{d\int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{A}$$

<u>Exercise 2</u> Find the center of mass of a semicircular plate of uniform density of radius r.

<u>Exercise 3</u> (Example 1/6.6) The triangular plate shown has a constant density of $d = 3g/cm^2$.

(a) Find the plate's moments about the axes.
(b) Find the plate's mass.
(c) Find the coordinates of the center of mass.



<u>Exercise 4</u> (Example 2/6.6) Find the center of mass of a thin plate covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x-axis. Assume the density of the plate at the point (x, y) is $d = 2x^2$.

6. Plates Bounded by Two Curves

Suppose a plate covers a region R that lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ and $a \leq x \leq b$.

- The center of mass of R is given by the coordinates:

$$\overline{x} = \frac{M_y}{m} = \frac{\int_a^b d\,x[f(x) - g(x)]dx}{m};$$

if the density is constant, then

$$\overline{x} = \frac{d\int_a^b x[f(x) - g(x)]dx}{d\int_a^b [f(x) - g(x)]dx} = \frac{\int_a^b x[f(x) - g(x)]dx}{\int_a^b [f(x) - g(x)]dx}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\int_a^b d\,\frac{1}{2}\left[f^2(x) - g^2(x)\right]dx}{m};$$

if the density if constant, then

$$\overline{y} = \frac{d\int_a^b \frac{1}{2}\left[f^2(x) - g^2(x)\right]dx}{d\int_a^b [f(x) - g(x)]dx} = \frac{\int_a^b \frac{1}{2}\left[f^2(x) - g^2(x)\right]dx}{\int_a^b [f(x) - g(x)]dx}$$

<u>Exercise 5</u> (Exercise 4/6.6) Find the center of mass of a thin plate of constant density covering the region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$.

<u>Exercise 6</u> (Exercise 17/6.6) The region bounded by the curves $y = \pm\frac{4}{\sqrt{x}}$ and the lines $x = 1$ and $x = 4$ is revolved about the y-axis to generate a solid.

a) Find the volume of the solid.

b) Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $d(x) = \frac{1}{x}$.

c) Sketch the plate and show the center of mass in your sketch.

<u>Exercise 8</u> (Exercise 32/6.6) Find the centroid of the thin plate bounded by $g(x) = 0,\ f(x) = 2 + \sin x,\ x = 0$ and $x = 2p$.

The Theorems of Pappus

Theorem 1 **Pappu's Theorem for Volumes**
If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If d is the distance from the axis of revolution to the centroid, then

$$V = 2p\,dA\,.$$

Theorem 2 **Pappu's Theorem for Surface Areas**
If an arc of a smooth plane curve is revolved once about a line that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length L of the arc times the distance traveled by the arc's centroid during the revolution. If d is the distance from the axis of revolution to the centroid, then

$$S = 2p\,dL\,.$$

<u>Exercise 9</u> (Exerise 35/6.6) The square region with vertices (0,2), (2,0), (4,2) and (2,4) is revolved about the x-axis to generate a solid. Find the volume and surface area of the solid.