10.7 Power Series

Definition 1 A **power series about** $x=0$ is a series of the form

$$\text{(1)} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

where *x* is a variable and the c_n's are constants called the coefficients of the series.

For each fixed *x*, the series (1) is a series of constants that we can test for convergence or divergence. A power series may converge for some values of *x* and diverge for other values of *x*.

The sum of series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

whose domain is the set of all x for which the series converges.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Definition 2 More generally, a series of the form

$$\text{(2)} \qquad \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

is called a **power series about** $x=a$ or a **power series centered at** $\boldsymbol{a}$**.**

Notes: 1. In writing out the term corresponding to $n=0$ in equations (1) and (2) we have adopted the convention that $(x-a)^n = 1$ even when $x=a$.

2. When $x=a$ all of the terms are 0 for $n \geq 1$ and so the power series (2) always converges when $x=a$.

Exercise 1 For what values of *x* is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

Exercise 2 For what values of *x* is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

The main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry. In particular, the sum of the power series in the next example is called a Bessel function, after the German astronomer Friedrich Bessel (1784-1846). These functions (now called Bessel functions) first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, these functions have been applied in may different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead.

Exercise 3 Find the domain of the Bessel function of order 0 defined by $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$.

Theorem - The Convergence Theorem for Power Series (10.7)

> If the power series
> $$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$
> converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$.
> If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ behaves in one of three possible ways. It might converge only at $x = a$, or converge everywhere, or converge on some interval of radius R centered at $x = a$. The number R is called the **radius of convergence** of the power series, and the interval centered at $x = a$ is called the **interval of convergence**.

Corollary (10.7)

> For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:
>
> 1. The series **converges** at $x = a$ and **diverges elsewhere** ($R = 0$).
> 2. The series **converges absolutely** for every x ($R = \infty$).
> 3. There is a positive number R such that the series **converges absolutely** if $|x-a| < R$ and the series **diverges** if $|x-a| > R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$

How to Test a Power Series for Convergence

1. Use the Ratio Test or Root Test to find the interval where the series converges absolutely. Ordinarily, this is an open interval $|x-a| < R$.
2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use a Comparison Test, the Integral test, or the Alternating Series Test.

Exercise 4 Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Exercise 5 Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

Representations of Functions as Power Series and Operations on Power Series

We will learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. This strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials. Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.

We have seen that

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x}, \ |x| < 1.$$

Here our point of view is different. We now say that

$$(3) \qquad \frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n \ , \ |x| < 1.$$

We now regard Equation (3) as expressing the function $f(x) = \dfrac{1}{1-x}$ as a sum of power series.

Exercise 6 Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Exercise 7 Find a power series representation for $\dfrac{1}{x+2}$.

Exercise 8 Find a power series representation of $\dfrac{x^3}{x+2}$.

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ whose domain is the interval of convergence of the series.

We would like to be able to differentiate and integrate such functions, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial.

<u>Theorem – The Term-by-Term Differentiation and Integration Theorem (10.7)</u>

If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$. This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

(i) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

Also

(2) $\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1} + C$ for $a - R < x < a + R$.

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Notes: Although the above Theorem says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the interval of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

Exercise 9 Express $\dfrac{1}{(1-x)^2}$ as a power series by differentiating Equation (3).

Exercise 10 Find a power series representation for $\ln(1-x)$ and its radius of convergence.

Exercise 11 Find a power series representation for $f(x) = \tan^{-1} x$.

Theorem – The Series Multiplication Theorem for Power Series (10.7)

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)\,B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Theorem (10.7)

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n \left(f(x)\right)^n$ converges absolutely for any continuous function f on $|f(x)| < R$