10.6
Alternating Series, Absolute and Conditional Convergence

Definition1 An alternating series is a series whose terms are alternately positive and negative, in other words, for which $a_n a_{n+1} < 0$, for any n.

An alternating series is a series of the form $\sum_{n=1}^{\infty} (-1)^n u_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$, where $u_n = |a_n|$.

Examples $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - ... = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

Theorem - The Alternating Series Test (Leibniz's Test)

If (a_n) is a sequence of positive terms ($a_n > 0$ for any n) such that

a) $a_n \ge a_{n+1}$ for all n (or starting with an index n_0)
b) $\lim_{n \to \infty} a_n = 0$

then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Exercise 1 Are the following alternating series convergent ?

a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ b) $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1}$

c) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

Theorem – The Alternating Series Estimation Theorem

If $s = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is the sum of an alternating series that satisfies the conditions from Leibniz's Test

$a_n \ge a_{n+1}$ for all n (or starting with an index n_0) and $\lim_{n \to \infty} a_n = 0$

then the error $R_n = s - s_n$ satisfies $|R_n| = |s - s_n| \le a_{n+1}$ for any n and R_n has the same sign as the first unused term .

The Theorem says that for the series that satisfy the conditions of the Leibniz's Test, the size of the error is smaller than a_{n+1}, which is the absolute value of the first neglected term.

Absolute and Conditional Convergence

We saw in Exercise 1a) that the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

is convergent. If we consider the series

$$|1|+\left|-\frac{1}{2}\right|+\left|\frac{1}{3}\right|+\left|-\frac{1}{4}\right|+\ldots=\sum_{n=1}^{\infty}\frac{1}{n}$$

which is divergent (p-series with p=1).

We see that $\sum_{n=1}^{\infty} a_n$ is convergent while $\sum_{n=1}^{\infty} |a_n|$ is divergent.

<u>Definition</u> Let $\sum_{n=1}^{\infty} a_n$ a series.

1. We say that the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** (AC) if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.
2. We say that the series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** (or converges conditionally) (CC) if $\sum_{n=1}^{\infty} a_n$ is convergent while $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Examples $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally

<u>Theorem – The Absolute Convergence Test</u>

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
(If a series is absolutely convergent, then it is convergent.)

Notes

1. The Converse of the Absolute Convergence Test is not true. There are convergent series that are not absolutely convergent, for example $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.
2. For the series with positive terms, the notions of convergence and absolute convergence coincide.
3. To decide the nature of the series $\sum_{n=1}^{\infty} |a_n|$ we can apply all the convergence tests learned for series with positive terms.
4. As we have seen, if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. In general, we will not be able to decide the nature of $\sum_{n=1}^{\infty} a_n$ if $\sum_{n=1}^{\infty} |a_n|$ diverges.

   However, if the series $\sum_{n=1}^{\infty} |a_n|$ is **divergent by the Ratio Test or Root Test**, then the series $\sum_{n=1}^{\infty} a_n$ **diverges as well** (as $\lim_{n\to\infty} a_n \neq 0$).

Exercise 2 Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Exercise 3 Determine whether $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

Theorem – The Rearrangement for Absolute Convergent Series

> If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots b_n, \ldots$ is any arrangement of the sequence (a_n), then $\sum_{n=1}^{\infty} b_n$ converges absolutely and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.

Notes

1. We cannot rearrange the terms of a conditionally convergent series and expect the new series to be the same as the original one. When we are using a conditionally convergent series, the terms must be added together in the order they are given to obtain a correct result.

2. The above theorem guarantees that the terms of an absolutely convergent series can be summed in any order without affecting the result.