10.3, 10.4, and 10.5
Series with nonnegative terms

In these sections we will study series of nonnegative terms

$$\sum_{n=1}^{\infty} a_n \text{ , where } a_n \ge 0, \forall n \in N$$

Recall that

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ converges if } p > 1 \text{ and diverges if } p \le 1$$

Theorem - The Integral Test (10.3)

> Let f be a continuous, positive, and decreasing function on $[1,\infty)$ with $f(n) = a_n$ for any *n*.
>
> Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.
>
> (in other words, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ both converge or both diverge)

Note: Theorem is also true if instead of $[1,\infty)$ we have $[n_0, \infty)$, where $f(n) = a_n$ for any $n \ge n_0$.

Then, $\sum_{n=n_0}^{\infty} a_n$ and $\int_{n_0}^{\infty} f(x)dx$ are both convergent or both divergent.

Exercise 1 – The *p*-series For what values of *p* (real number) is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

(Example 3.10.3)

Solution

Exercise 2 Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n+4}$ converges or diverges.

(#4,/10.3)

Exercise 3 Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Estimating the sum of a series

Suppose we have been able to use the Integral Test to show a series $\sum_{n=1}^{\infty} a_n$ is convergent and now we want to find an approximation to the sum s of the series.

Any partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$. But how good is such an approximation?

To find out, we need to estimate the size of the remainder .

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \ldots$$

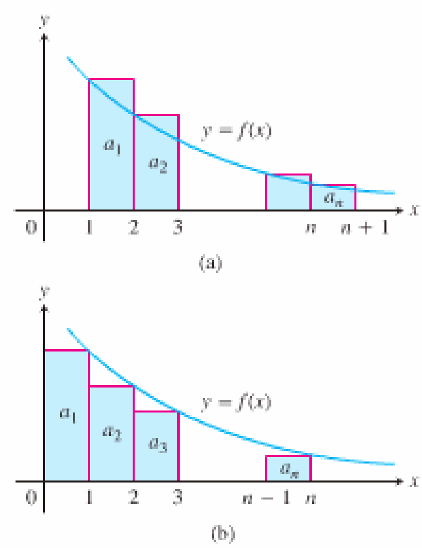

FIGURE 10.11 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

From figure (a), we see that $a_{n+1} + a_{n+2} + \ldots \geq \int_{n+1}^{\infty} f(x)\,dx$.

From figure (b), we see that $a_{n+1} + a_{n+2} + \ldots \leq \int_{n}^{\infty} f(x)\,dx$.

Therefore,

$$\boxed{\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx}$$

The Remainder Estimate for the Integral Test

Exercise 4 a) Approximate the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
b) How many terms are required to ensure that the sum is accurate to within 0.0005?

<u>Note</u>

$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$, then

$\int_{n+1}^{\infty} f(x)\,dx \le s - s_n \le \int_{n}^{\infty} f(x)\,dx$, then

$$\boxed{s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_{n}^{\infty} f(x)\,dx}$$

, which gives a lower bound and upper bound for s.

This is a more accurate approximation to the sum of the series than the partial sum s_n.

Exercise 5 Estimate the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ using $n = 10$ in the above formula.

<u>Property 1 (10.3)</u>

The series $\sum_{n=1}^{\infty} a_n$ with $a_n \ge 0$ converges if and only if the sequence of its partial sums (s_n) is bounded from above.

Proof

Theorem – The Comparison Test (10.4)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $a_n \geq 0$, $b_n \geq 0$, and $a_n \leq b_n$ for all n (or starting with an index n_0).

(1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof

Exercise 6 Determine whether $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Exercise 7 Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergences.

Theorem – The Limit Comparison Test (10.4)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that $a_n > 0$, $b_n > 0$ for all n (or starting with an index n_0).

(1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ ($c \in \mathbb{R}$), then the two series both converge or both diverge.

(2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and

if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Exercise 8 Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergences.

Exercise 9 Test the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ for convergence or divergences.

Theorem – The Root Test (10.5)

Let $\sum_{n=1}^{\infty} a_n$ with $a_n \ge 0$ and $\lim_{n\to\infty} \sqrt[n]{a_n} = r$.

(1) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $r > 1$ $(1 < r \le \infty)$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note If $r = 1$ the test is inconclusive.

Theorem – The Ratio Test (10.5)

Let $\sum_{n=1}^{\infty} a_n$ with $a_n > 0$. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then

(1) If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $l > 1$ $(1 < l \le \infty)$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note If $l = 1$ the test is inconclusive.

Exercise 10 Use the Ratio Test to determine if the series converges or diverges.

(#2/10.5) $$\sum_{1}^{\infty} \frac{n+2}{3^n}$$

Exercise 11 Use the Root Test to determine if the series converges or diverges.

(#10/10.5) $$\sum_{1}^{\infty} \frac{4^n}{(3n)^n}$$

Exercise 12 Determine the nature of the series $\sum_{n=1}^{\infty} \frac{n}{2n^2 + 1}$

<u>Notes</u> 1. If we can find the nature of a series $\sum a_n$ using the **Ratio Test**, then we can also find its nature using the **Root Test.**

2. If the Root Test does not give any information about the convergence or divergence of the series $\sum a_n$, then neither does the Ratio Test.

3. If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$, then $\lim_{n\to\infty}\sqrt[n]{a_n}=1$, and **neither the Ratio Test nor the Root Test** will give any information about the convergence or divergence of the series. In this case it is recommended to try to use a comparison test.

More Exercises 10.5

Determine if each series converges or diverges:

#18 $\sum_{n=1}^{\infty} n^2 e^{-n}$

#22 $\sum_{n=1}^{\infty}\left(\frac{n-2}{n}\right)^n$

#23 $\sum_{n=1}^{\infty}\frac{2+(-1)^n}{1.25^n}$

#27 $\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$

#28 $\sum_{n=1}^{\infty}\frac{(\ln n)^n}{n^n}$

#29 $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$

#32 $\sum_{n=1}^{\infty}\frac{n\ln n}{2^n}$

#34 $\sum_{n=1}^{\infty} e^{-n}\left(n^3\right)$

#40 $\sum_{n=2}^{\infty}\frac{n}{(\ln n)^{\frac{n}{2}}}$

#42 $\sum_{n=1}^{\infty}\frac{3^n}{n^3 2^n}$

#44 $\sum_{n=1}^{\infty}\frac{(2n+3)\left(2^n+3\right)}{3^n+2}$

#45 $a_1=2, a_{n+1}=\frac{1+\sin n}{n}a_n$

#48 $a_1=3, a_{n+1}=\frac{n}{n+1}a_n$

#55 $\sum_{n=1}^{\infty}\frac{2^n n!n!}{(2n)!}$

#58 $\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{n^2}}$

#60 $\sum_{n=1}^{\infty}\frac{n^n}{\left(2^n\right)^2}$

Answers: 18) C; 22) D); 23) C; 27) C; 28) C; 29) D; 31) D; 32) C; 34) C; 40)C; 41) C; 42) D; 44) C; 45) C; 48) D; 55) C; 58) C; 60) D