10.2 Infinite Series

<u>Definition 1</u> Given an infinite sequence $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

is an **infinite series**.

<u>Definition 2</u> Given a series $\sum_{n=1}^{\infty} a_n$, the sums of the form

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$

and, in general,

$$s_n = a_1 + a_2 + \ldots + a_n$$

are called the **partial sums** of the series.

<u>Definition 3</u> Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, let s_n denote its nth partial sum:

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is called **convergent** and we write

$$\sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

<u>Note</u>: The sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

<u>Definition 4</u> A series of the form

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$ is called a **geometric series.**
(Each term is obtained from the preceding one by multiplying it by the common ration r)

When is a geometric series convergent?

Exercise 1 Show that $2.3\overline{17} = 2.3171717...$ is a rational number (a ratio of integers).

Exercise 2 Find the sum of the series $\sum_{n=1}^{\infty} x^n$, where $|x| < 1$.

<u>Theorem</u>

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof

<u>Note</u> With any series $\sum a_n$ we associate two sequences: the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\sum a_n$ is convergent, then $\lim_{n\to\infty} s_n = s$ (the sum of the series) and, according to the above Theorem, $\lim_{n\to\infty} a_n = 0$.

<u>Note</u> The converse of the above Theorem is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum a_n$ is convergent.

<u>Theorem (The nth-Term Test for Divergence)</u>

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Exercise 3 Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

<u>Theorem</u>

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum k a_n$ (where k is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

1. $\sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
3. $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

<u>Corollaries</u>

1. If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} k a_n$ is divergent.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} (a_n - b_n)$ both diverge.

More Exercises 10.1

Which of the sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

#44/10.1 $a_n = np\cos(np)$ **#46/10.1** $a_n = \dfrac{\sin^2 n}{2^n}$ **#48/10.1** $a_n = \dfrac{3^n}{n^3}$

#54/10.1 $a_n = \left(1 - \dfrac{1}{n}\right)^n$ **#55/10.1** $a_n = \sqrt[n]{10n}$ **#65/10.1** $a_n = \dfrac{n!}{10^{6n}}$

#88/10.1 $a_n = \dfrac{1}{\sqrt{n^2-1} - \sqrt{n^2+n}}$ **#90/10.1** $a_n = \displaystyle\int_1^n \frac{1}{x^p}\,dx,\ p > 1$

#92/10.1 Assume that the sequence converges. Find its limit.

$$a_1 = -1,\ \ a_{n+1} = \frac{a_n + 6}{a_n + 2}$$

#112/10.1 Determine if the sequence is monotonic and if it is bounded.

$$a_n = \frac{(2n+3)!}{(n+1)!}$$

More Exercises 10.2

Use the nth-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

#28/10.2 $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$ **#30/10.2** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

#36/10.2 Find a formula for the nth partial sum of the series and use it to determine if the series converges or diverges. If the series converges, find its sum.

$$\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2} \right)$$

#42/10.2 Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$.

Which series converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

#51/10.2	$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{2^n}$	**#52/10.2**	$\sum_{n=1}^{\infty}(-1)^{n+1}n$	**#58/10.2**	$\sum_{n=1}^{\infty}\frac{1}{x^n}, \lvert x\rvert>1$
#64/10.2	$\sum_{n=1}^{\infty}\frac{2^n+4^n}{3^n+4^n}$	**#65/10.2**	$\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$	**#68/10.2**	$\sum_{n=1}^{\infty}\frac{e^{np}}{p^{ne}}$

#70/10.2 Write out the first few terms of the series to find *a* and *r*, and find the sum of the series. Then express the inequality $|r|<1$ in terms of *x* and find the values of *x* for which the inequality holds and the series converges.

$$\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

#76/10.2 Find the value of *x* for which the given geometric series converges. Also, find the sum of the series for those values of *x*.

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(x-3)^n$$

Exercise 1 If the *n*th partial sum of a series $\sum_{n=1}^{\infty}a_n$ is $s_n=\frac{n-1}{n+1}$, find a_n and $\sum_{n=1}^{\infty}a_n$.

Exercise 2 A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0<r<1$. Suppose that the ball is dropped from an initial height of H meters.

a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.
b) Calculate the total time that the ball travels.

(we know that a ball falls $h=\frac{1}{2}gt^2$ meters in *t* seconds, where g is the gravitational acceleration.)

Exercise 3 A right triangle at C, ABC is given with $\angle A=q$ and $|AC|=b$. CD is drawn perpendicular to AB, De is drawn perpendicular to BC, EF is drawn perpendicular to AB, and this process is continued indefinitely. Find the total length of all the perpendiculars

$$|CD|+|DE|+|EF|+|FG|+\ldots$$

Exercise 4 The Sierpinski carpet. A carpet is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.