10.1 Sequences

A sequence is a list of numbers written in a given order:

$$a_1, a_2, \ldots a_n, \ldots$$

Definition 1 **A sequence of real numbers** is a function

$$f : \mathbb{N} \to \mathbb{R}$$
$$f(n) = a_n, \forall n \in \mathbb{N}$$

Examples of sequences

- Some sequences can be defines by giving a formula for the nth term

  1) $\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$

  2) $\left(\sqrt{n-3}\right)_{n=3}^{\infty}$

- Some sequences don't have a simple defining equation

  3) $\{p_n\}$, where p_n is the population of the world on January 1^{st} in the year n

  4) $\{a_n\}$, where a_n is the digit in the nth decimal place of the number e

- Some sequences can be defined recursively

  5) The Fibonacci sequence $\{f_n\}$, where $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$, for any $n \geq 3$

  This sequence arose when the Italian mathematician (13^{th} century) solved a problem concerning the breeding of rabbits.

Definition 2 **Bounded Sequences**

- A sequence (a_n) is **bounded above** if and only if

there is $M \in \mathbb{R}$ such that $a_n \le M, \forall n$

- A sequence (a_n) is **bounded below** if and only if

there is $m \in \mathbb{R}$ such that $a_n \ge m, \forall n$

- A sequence (a_n) is **bounded** if and only if it is bounded below and above , that is iff

there is $m \in \mathbb{R}$ and $M \in \mathbb{R}$ such that $m \le a_n \le M, \forall n$

- A sequence (a_n) is **unbounded** if it is not bounded, that is, iff

outside any closed interval $[a, b]$ there is at least one term of the sequence.

Note that a sequence (a_n) is not bounded if it is not bounded above or it is not bounded below or both.

Examples

1) $x_n = (-1)^n$

2) $a_n = \dfrac{1}{n}$

3) $b_n = 2^n$

4) $c_n = -n$

5) $y_n = (-1)^{n-1} n$

Definition 3 **Increasing/Decreasing Sequences**

- A sequence (a_n) is **increasing** if and only if $a_n < a_{n+1}, \forall n \ge 1$.
- A sequence (a_n) is **decreasing** if and only if $a_n > a_{n+1}, \forall n \ge 1$.
- A sequence (a_n) is **nondecreasing** if and only if $a_n \le a_{n+1}, \forall n \ge 1$.
- A sequence (a_n) is **nonincreasing** if and only if $a_n \ge a_{n+1}, \forall n \ge 1$.
- A sequence is monotonic if and only if it is either nondecreasing or nonincreasing.

Note: Any increasing sequence is nondecreasing.
Any decreasing sequence is nonincreasing.

Examples

1) $a_n = 1 - \dfrac{1}{n}$

2) $b_n = \dfrac{1}{n^2}$

3) $c_n = (-1)^n$

The Limit of a Sequence

Definition 4 **A sequence (a_n) has the limit $L \in \mathbb{R}$** if and only if any neighborhood of L contains all the terms, except, maybe, a finite number of them.

In other words, $\lim\limits_{n\to\infty} a_n = L$ if and only if we can make the terms a_n as close to L as we like by taking n sufficiently large.

Definition 5 **A sequence (a_n) is convergent** if and only if $\lim\limits_{n\to\infty} a_n = L \in \mathbb{R}$.

A sequence (a_n) is divergent if and only if $\lim\limits_{n\to\infty} a_n$ does not exist (in $\mathbb{R}$).

Property

$\lim\limits_{n\to\infty} a_n = L$ if and only if $\lim\limits_{n\to\infty} |a_n - L| = 0$.

Examples

1) $\left(\dfrac{1}{n}\right)_{n=1}^{\infty}$

2) $x_n = (-1)^n$

Theorem

The limit of a sequence is unique.

Theorems

By **adding or eliminating a finite number of terms**,

a) a convergent sequence remains convergent
b) a divergent sequence remains divergent.

By **changing the order of the terms**

a) of a convergent sequence we obtain another convergent sequence with the same limit.
b) of a divergent sequence we obtain another divergent sequence.

Theorems

Any convergent sequence is bounded.
Any nonbounded sequence is divergent.

Examples

1) $a_n = n$

2) $b_n = -n$

3) $c_n = (-1)^n$

Theorem (The Monotonic Sequence Theorem)

If (a_n) is bounded and monotonic, then (a_n) is convergent.

Operations with Convergent Sequences

Theorem

Let $(a_n),(b_n)$ sequences of real numbers and $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Then:

1) $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$

2) $\lim_{n\to\infty}(ka_n) = ka$, $\forall k \in \mathbb{R}$

3) $\lim_{n\to\infty}(a_n b_n) = ab$

4) $\lim_{n\to\infty}\frac{1}{a_n} = \frac{1}{a}$ if $a \neq 0$

5) $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$

6) $\lim_{n\to\infty} a_n{}^p = a^p$ if $p > 0, a_n > 0$.

Theorem (The Sandwich Theorem for Sequences)

If (a_n), (b_n), and (c_n) are sequences of real numbers and

$a_n \le b_n \le c_n$, for any $n \ge n_0$ (beyond some index n_0) and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \in \mathbb{R}$$

then

$$\lim_{n\to\infty} b_n = L .$$

Corollary

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Proof

Theorem (The Continuous Function Theorem for Sequences)

If (a_n) is a sequence of real numbers and $\lim_{n\to\infty} a_n = L$, where L is a real number and f is a continuous function at L, defined for all a_n, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Theorem (about using l'Hopital's Rule)

If f is a function defined for all $x \ge n_0$,

(a_n) is a sequence of real numbers,

$f(n) = a_n$, for any $n \ge n_0$ and

$\lim_{x\to\infty} f(x) = L$,

then $\lim_{n\to\infty} a_n = L$.

Exercise 1 Find $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

Theorem

Let (a_n) be a sequence of real numbers.

1) If $(u_n)_{n\geq 0}$, $\lim\limits_{n\to\infty} u_n = \infty$, and $a_n \geq u_n, \forall n \geq n_0$ (n_0 fixed), then $\lim\limits_{n\to\infty} a_n = \infty$.

2) If $(v_n)_{n\geq 0}$, $\lim\limits_{n\to\infty} v_n = -\infty$, and $a_n \leq v_n, \forall n \geq n_0$ (n_0 fixed), then $\lim\limits_{n\to\infty} a_n = -\infty$.

Theorem

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ sequences of real numbers.

1) If $a_n \to \infty$ and $b_n \to b$, where $b \in \mathbb{R}$, then $\lim\limits_{n\to\infty}(a_n + b_n) = \infty$

$$\lim_{n\to\infty}(a_n b_n) = \begin{cases} \infty & \text{if } b > 0 \\ -\infty & \text{if } b < 0 \end{cases}$$

2) If $a_n \to -\infty$ and $b_n \to b$, where $b \in \mathbb{R}$, then $\lim\limits_{n\to\infty}(a_n + b_n) = -\infty$

$$\lim_{n\to\infty}(a_n b_n) = \begin{cases} -\infty & \text{if } b > 0 \\ \infty & \text{if } b < 0 \end{cases}$$

3) If $a_n \to \infty$ and $b_n \to \infty$, then $\lim\limits_{n\to\infty}(a_n + b_n) = \infty$

$$\lim_{n\to\infty}(a_n b_n) = \infty \quad .$$

4) If $a_n \to -\infty$ and $b_n \to -\infty$, then $\lim\limits_{n\to\infty}(a_n + b_n) = -\infty$

$$\lim_{n\to\infty}(a_n b_n) = \infty .$$

5) If $a_n \to \infty$ and $b_n \to -\infty$, then $\lim\limits_{n\to\infty}(a_n b_n) = -\infty$.

Indeterminate Cases $\infty - \infty, \infty \cdot 0, \dfrac{0}{0}, \dfrac{\infty}{\infty}, 0^0, \infty^0, 1^\infty$

Theorem

If $a_n \to \infty$, then $\dfrac{1}{a_n} \to 0$.

If $a_n \to -\infty$, then $\dfrac{1}{a_n} \to 0$.

If $b_n \to 0$ and $b_n > 0$ for any $n \geq n_0$, then $\dfrac{1}{b_n} \to \infty$.

If $b_n \to 0$ and $b_n < 0$ for any $n \geq n_0$, then $\dfrac{1}{b_n} \to -\infty$.