Math 180

-1-

SECTION 4.4

(11) $f(x) = x^3 - 3x + 3$

x	$-\infty$	$x_1 \approx -2.1$	-1	0	1	∞
f'	$+ \ + \ +$	$+$	0	$- \ -$	$- \ 0$	$+ \ + \ +$
f	$-\infty \to$	$0 \to$	$5 \searrow$	$3 \searrow$	$1 \nearrow$	∞
f''	$- \ - \ -$	$- \ -$	$-$	0	$+ \ + \ +$	$+$

(f) Domain: $x \in \mathbb{R}$

x-n: $f(x) = 0$

$x^3 - 3x + 3 = 0 \Rightarrow x_1 \simeq -2.1$

y-n: $x = 0,\ y = 3$

$\lim\limits_{\infty} f(x) = \lim\limits_{x \to \infty} x^3\left(1 - \frac{3}{x^2} + \frac{3}{x^3}\right) = \infty$

$\lim\limits_{-\infty} f(x) = -\infty$

(f') $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$

$f'(x) = 0 \Rightarrow x = \pm 1$

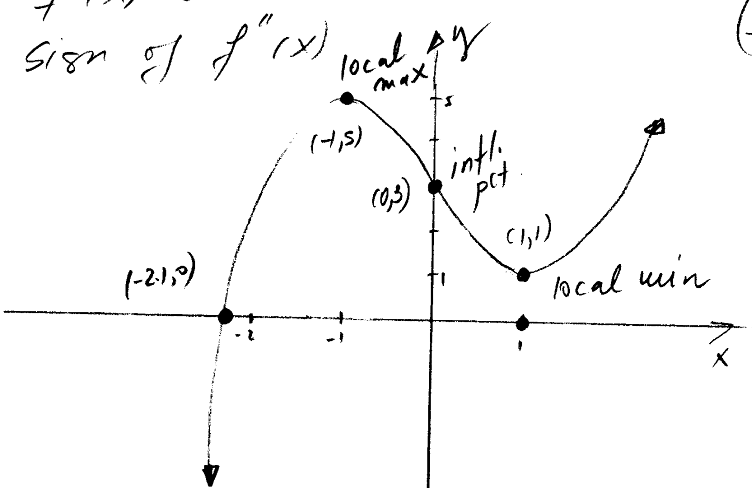
Sign of $f'(x)$: $+ \quad -1 \quad - \quad 1 \quad +$

$f(1) = 1,\ f(-1) = 5$

(f'') $f''(x) = 6x$

$f''(x) = 0 \Rightarrow x = 0$

Sign of $f''(x)$



(14) $y = (x-2)^3 + 1$

x	$-\infty$	0	1	2	∞
f'	$+ \ +$	$+$	$+ \ +$	0	$+ \ + \ +$
f	$-\infty \to$	$-7 \to$	$0 \to$	$1 \to$	∞
f''	$- \ -$	$-$	$- \ -$	0	$+ \ + \ +$

(f) Domain: $x \in \mathbb{R}$

x-n: $(x-2)^3 + 1 = 0$

$(x-2)^3 = -1 \quad / \sqrt[3]{\ }$

$x - 2 = \sqrt[3]{-1},\quad x - 2 = -1$

$x = 1$

y-n: $x = 0,\ y = -7$

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{\infty} x^3\left[\left(1 - \frac{2}{x}\right)^3 + \frac{1}{x^3}\right]$

$= \infty(1+0) = \infty$

$\lim\limits_{x \to -\infty} f(x) = -\infty(1+0) = -\infty$

(f') $f'(x) = 3(x-2)^2$

$f'(x) = 0$ when $x = 2$

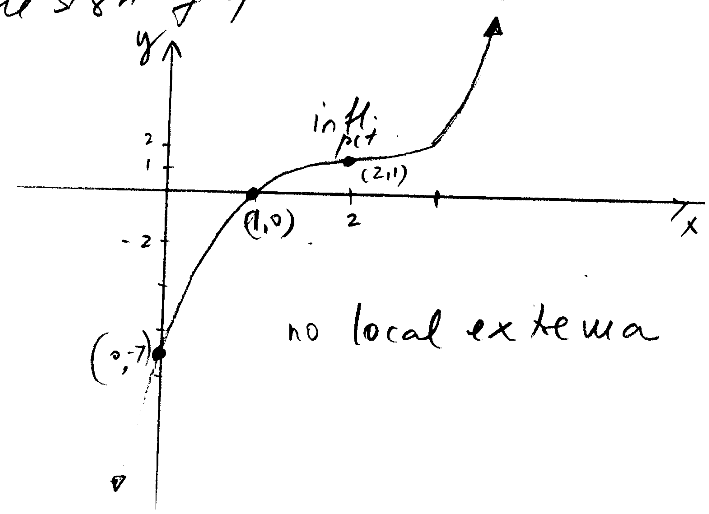
$f'(x) > 0$ for $\forall\, x \neq 2$

$f(2) = 1$

(f'') $f''(x) = 6(x-2)$

$f''(x) = 0$ when $x = 2$

The sign of f'' is given by $x - 2$



(39) $y = xe^{-x}$

x	$-\infty$			0	1	2	∞
f'	+ +	+	+	0	− −	−	−
f	$-\infty$ $\nearrow$		0 $\nearrow$		$\frac{1}{e}$ $\searrow$	$\frac{2}{e^2}$ $\searrow$	H.A $y=0$
f''	− −	− −	−	−	0	+	+

(f) Domain: $x \in \mathbb{R}$

x-n, y-n: $(0,0)$ $\quad (e^{-x} \neq 0)$

$$\lim_{\infty} f(x) = \lim_{\infty} xe^{-x} = \infty \cdot 0$$

$$= \lim_{\infty} \frac{x}{e^x} = \frac{\infty}{\infty} \quad (\text{l'Hopital})$$

$$= \lim_{\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

$$\lim_{-\infty} f(x) = \lim_{-\infty} xe^{-x} = -\infty \cdot (e^{\infty}) = -\infty \cdot \infty = -\infty$$

(f') $f'(x) = e^{-x} + x(-1)e^{-x}$

$f'(x) = e^{-x}(1-x)$

$f'(x) = 0$ when $x = 1$

The sign of f' is given by $1-x$ $\quad (e^{-x} > 0, \forall x)$

+ 1 − $\to x$

$f(1) = e^{-1} = \frac{1}{e}$

(f'') $f'(x) = e^{-x}(1-x)$

$f''(x) = -e^{-x}(1-x) + e^{-x}(-1)$

$f''(x) = -e^{-x}(1-x+1) = -e^{-x}(2-x)$

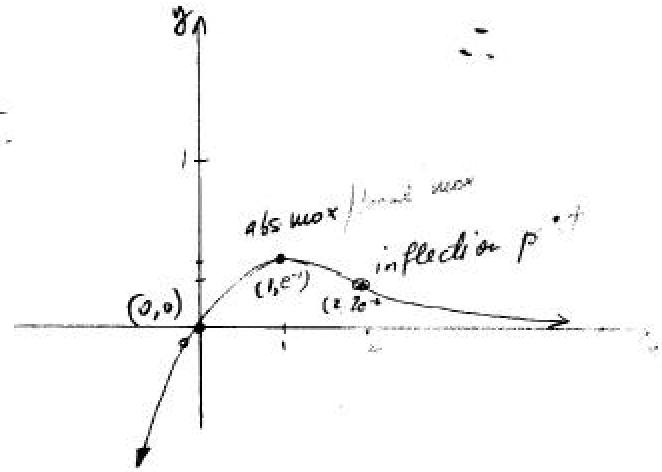
$f''(x) = e^{-x}(x-2)$

$f''(x) = 0$ when $x = 2$

$f(2) = 2e^{-2}$

The sign of f'' is given by $x-2$

$(e^{-x} > 0, \forall x)$



(41) $f(x) = \dfrac{1}{1+e^{-x}} \quad \dfrac{e^x}{e^x+1}$

x	$-\infty$			0			∞
f'	+ +	+	+	+	+	+	+
f	$y=0$ H.A	$\nearrow$		$\frac{1}{2}$	$\nearrow$		$y=1$ H.A
f''							

(f) Domain: $x \in \mathbb{R}$ $(e^x + 1 > 0, \forall x)$

x-n: $y \neq 0$ $\quad (e^x \neq 0)$

y-n: $x = 0$, $y = \frac{1}{2}$

$$\lim_{\infty} f(x) = \lim_{\infty} \frac{e^x}{e^x+1} = \frac{\infty}{\infty} \quad (\text{l'Hopital})$$

$$= \lim_{\infty} \frac{e^x}{e^x} = \lim_{\infty} 1 = 1 \Rightarrow \text{H.A. } y = 1 \ (x \to \infty)$$

$$\lim_{-\infty} f(x) = \lim_{-\infty} \frac{1}{1+e^{-x}} = \frac{1}{\infty} = 0 \Rightarrow \text{H.A. } y = 0 \ (x \to -\infty)$$

(f') $f(x) = \dfrac{e^x}{e^x+1} = \dfrac{e^x+1-1}{e^x+1} = 1 - \dfrac{1}{e^x+1}$

$f'(x) = \dfrac{e^x}{(e^x+1)^2} \qquad f'(x) \neq 0, \forall x$

$\qquad f'(x) > 0, \forall x$

(f'') $f'(x) = \dfrac{e^x}{(e^x+1)^2}$

$$f''(x) = \frac{e^x(e^x+1)^2 - e^x(2)(e^x+1)e^x}{(e^x+1)^4}$$

$$= \frac{e^x\cancel{(e^x+1)}(e^x+1-2e^x)}{(e^x+1)^{\cancel{4}\,3}}$$

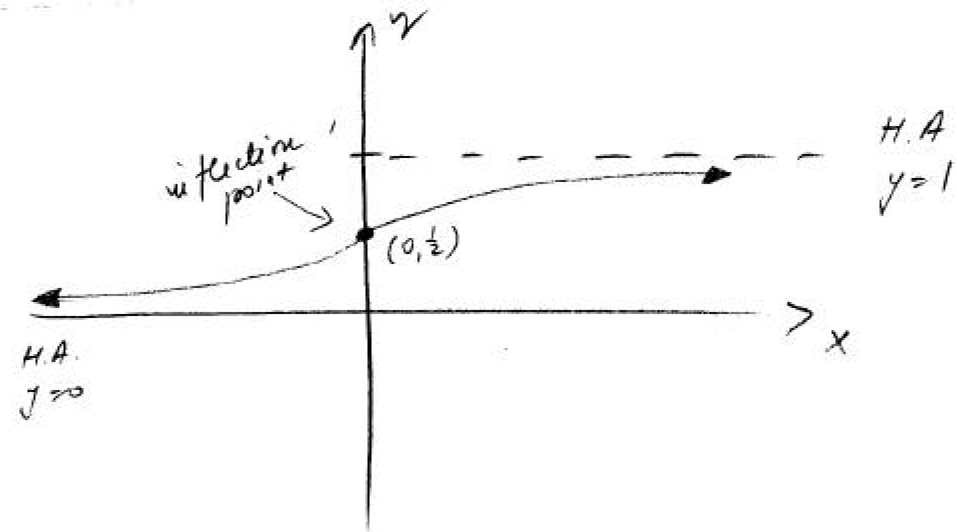
$$f''(x) = \frac{e^x(1-e^x)}{(1+e^x)^3}$$

$f''(x) = 0$ iff $1-e^x = 0$

$e^x = 1$

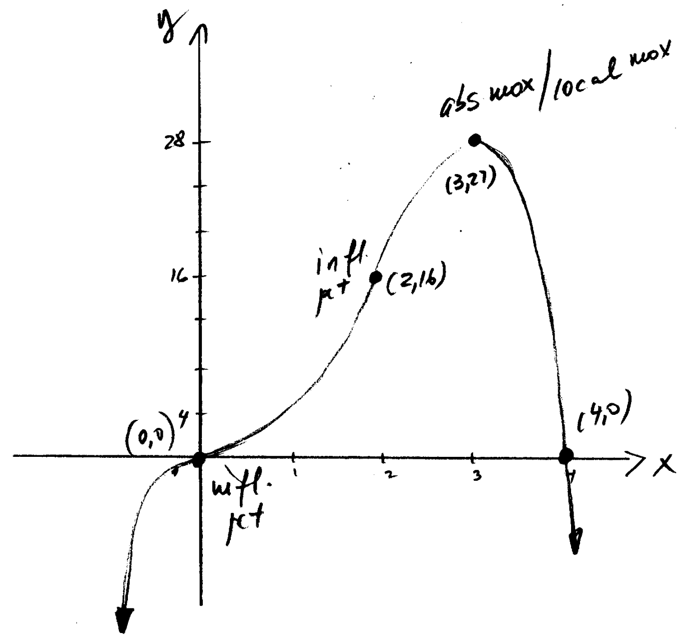
$x = 0$

The sign of $f''(x)$ is given by $1-e^x$



(17) $f(x) = 4x^3 - x^4$

x	$-\infty \quad 0 \quad 2 \quad 3 \quad 4 \quad \infty$
f'	$+ \quad + \quad 0 \quad + \quad + \quad 0 \quad - \quad - \quad -$
f	$-\infty \quad 0 \quad 16 \quad 27 \quad 0 \quad -\infty$
f''	$- \; - \; - \quad 0 \; + \; + \; 0 \quad - \; - \; - \; -$



(f) Domain: $x \in \mathbb{R}$

x-∩: $4x^3 - x^4 = 0$

$x^3(4-x) = 0 \; < \; x = 0, \; x = 4$

y-∩: $(0,0)$

$\lim_{\infty} f(x) = \lim_{\infty} x^3(4-x) = \infty(-\infty) = -\infty$

$\lim_{-\infty} f(x) = -\infty(\infty) = -\infty$

(f') $f(x) = 4x^3 - x^4$

$f'(x) = 12x^2 - 4x^3 = 4x^2(3-x)$

$f'(x) = 0 \; < \; x = 0, \; x = 3$

$f(3) = 27$

The sign of f' is given by $3 - x$

$(4x^2 > 0, \; \forall \; x \neq 0)$

(f'') $f'(x) = 12x^2 - 4x^3$

$f''(x) = 24x - 12x^2 = 12x(2-x)$

$f''(x) = 0 \; < \; x = 0, \; x = 2$

$f(2) = 16$

The sign of f'': $- \quad 0 \quad + \; + \quad 2 \quad - \; -$

(20) $f(x) = x\left(\frac{x}{2} - 5\right)^4$

x	$-\infty \quad 0 \quad 2 \quad 4 \quad 10 \quad \infty$
f'	$+ \; + \; + \; + \; 0 \; - \; - \; - \; - \; 0 \; + \; + \; +$
f	$-\infty \nearrow 0 \nearrow 512 \searrow 324 \searrow 0 \nearrow \infty$
f''	$- \; - \; - \; - \quad 0 \; + \; + \; 0 \; + \; +$

(f) Domain: $x \in \mathbb{R}$

x-∩: $f(x) = 0 \; < \; x = 0, \; \frac{x}{2} = 5, \; x = 10$

y∩: $(0,0)$

$\lim_{\infty} f(x) = \infty \cdot \infty = \infty$

$\lim_{-\infty} f(x) = -\infty(\infty) = -\infty$

(f') $f'(x) = \left(\frac{x}{2} - 5\right)^4 + x(4)\left(\frac{x}{2} - 5\right)^3 \cdot \frac{1}{2}$

$f'(x) = \left(\frac{x}{2} - 5\right)^4 + 2x\left(\frac{x}{2} - 5\right)^3$

$f'(x) = \left(\frac{x}{2} - 5\right)^3\left(\frac{x}{2} - 5 + 2x\right) = \left(\frac{x}{2} - 5\right)^3\left(\frac{5x}{2} - 5\right)$

$f'(x) = \frac{(x-10)^3}{8} \cdot \frac{5(x-2)}{2}$

$f'(x) = 0 <$ $x = 10$, $x = 2$

$f(2) = 2(-4)^4 = 512$

The sign of f' is given by $(x-10)(x-2)$ $\quad$ $+\ \ 2\ \ -\ \ 10\ \ +$

(f'') $f'(x) = \left(\frac{x}{2}-5\right)^3\left(\frac{5x}{2}-5\right)$

$$f''(x) = 3\left(\frac{x}{2}-5\right)^2 \cdot \frac{1}{2}\left(\frac{5x}{2}-5\right) + \left(\frac{x}{2}-5\right)^3 \cdot \frac{5}{2}$$

$$f''(x) = \frac{15}{2}\left(\frac{x}{2}-5\right)^2\left(\frac{x}{2}-1\right) + \frac{5}{2}\left(\frac{x}{2}-5\right)^3$$

$$= \frac{5}{2}\left(\frac{x}{2}-5\right)^2\left(3\cdot\left(\frac{x}{2}-1\right) + \left(\frac{x}{2}-5\right)\right)$$

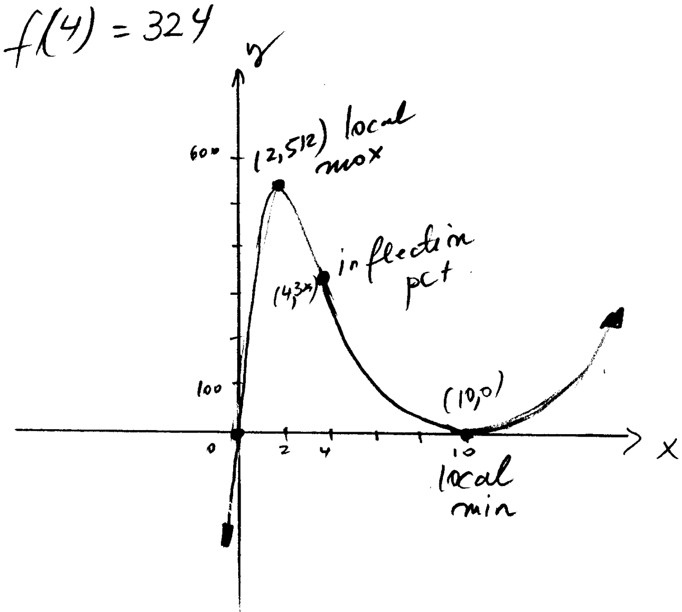
$$f''(x) = \frac{5}{2}\left(\frac{x}{2}-5\right)^2(2x-8)$$

$$f''(x) = 5\left(\frac{x}{2}-5\right)^2(x-4)$$

$f''(x) = 0 <$ $x = 10$, $x = 4$

The sign of f'' is given by $x-4$

$f(4) = 324$



(32) $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x<0 \\ \sqrt{x}, & x \geq 0 \end{cases}$

Note that $f(x) = \sqrt{|x|}$ is an even function ($f(-x) = f(x)$), so its graph is symmetric about y-axis. We'll do all the work on $[0,\infty)$, then reflect the graph about the y-axis.

$x \in [0, \infty)$

$f(x) = \sqrt{x}$

x	0						∞
f'	|	+	+	+	+	+	+ +
f	0	$\nearrow$					∞
f''	|	−	−	−	−	−	−

(f) intercepts: $(0,0)$

$\lim_{\infty} f(x) = \infty$

(f') $f'(x) = \frac{1}{2\sqrt{x}} > 0 \quad \forall x \neq 0$

(f'') $f''(x) = \frac{-2\cdot\frac{1}{2\sqrt{x}}}{4x} = \frac{-1}{4x\sqrt{x}} < 0 \quad \forall x \neq 0$