Sections 4.1 & 4.2

Some of the most important applications of differential calculus are optimization problems, in which we are required to find optimal (best) way of doing something.

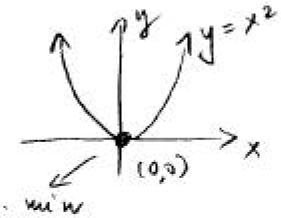
These problems can be reduced to finding the maximum or minimum values of a function.

Definition Let f be a function with domain D. A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in D. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in D. The number $f(c)$ is called the **minimum value** of f on D.

Note

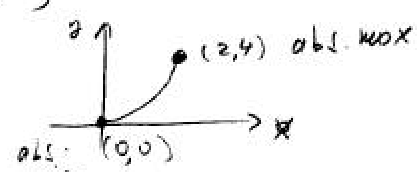
- The maximum and minimum values of f are called the **extreme values** of f.

Examples



(1) $f(x) = x^2, \; x \in \mathbb{R}$

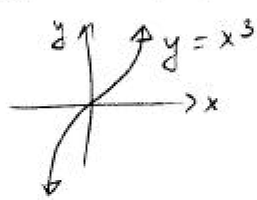
f has an absolute min. at $x = 0$ because $f(x) \geq f(0) \;\; \forall x \in \mathbb{R}$; The absolute minimum is $f(0) = 0$



(2) $f(x) = x^2, \; x \in [0,2]$

f has an absolute min. at $x = 0$, $f(0) = 0$

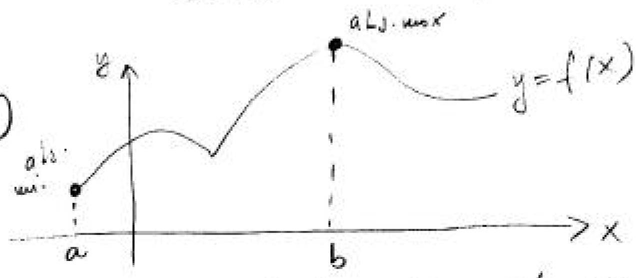
f has an absolute max. at $x = 2$, $f(2) = 4$



(3) $f(x) = x^3, \; x \in \mathbb{R}$

f has no absolute extrema

(4)



f has an absolute min. at $x = a$, $f(a)$

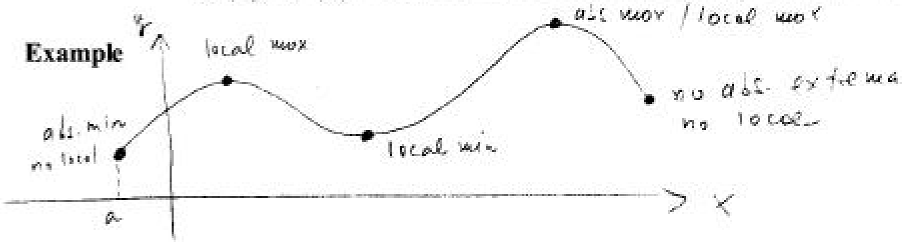
f has an absolute max. at $x = b$, $f(b)$

Note that $(a, f(a))$ = lowest point on the graph

$(b, f(b))$ = highest point on the graph

Definition A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c. (This means $f(c) \geq f(x)$ for all x in some open interval containing c.) The number $f(c)$ is called a **local maximum value** of f. Similarly, f has a **local minimum** (or **relative minimum**) at c if $f(c) \leq f(x)$ when x is near c. The number $f(c)$ is called a **local minimum value** of f.

Example



Fermat's Theorem (The First Derivative Theorem for Local Extreme Values) (4.1)

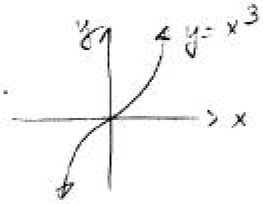
This theorem says that a function's derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined.

Hypothesis:	f is a function c is an interior point of the domain of f f has a local minimum or maximum value at c $f'(c)$ exists
Conclusion:	$f'(c) = 0$

Notes:

- Fermat's Theorem says that a function's derivative is always zero at an *interior point* where the function has a *local extreme value* and the *derivative is defined*.
- The Converse of the Fermat's Theorem is false in general:

  If $f'(c) = 0$, c is not necessarily a maximum or minimum .

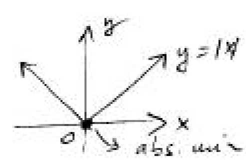


  - Example $f(x) = x^3, \ x \in \mathbb{R}$

    $f'(x) = 3x^2$

    $f'(0) = 0$, but $x = 0$ is not a local min/max

- There may be an extreme value where $f'(c)$ is not defined.



  - Example $f(x) = |x|$

    $f'(0)$ = does not exist

    but $f(0) = 0$ is an extrema point (abs. minimum)

- The only places where a function f can possibly have an extreme value (local or global) are:
  - interior points where $f' = 0$,
  - interior points where f' does not exist
  - endpoints of the domain of the function.

Note

- The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from its derivative.

Corollaries of the Mean Value Theorem

Corollary 1 Functions with Zero Derivatives Are Constant.

Corollary 2 Functions with the Same Derivative Differ by a Constant.

<u>Corollary 1</u>

Given $f'(x) = 0, \forall x \in (a,b)$

Prove $f(x) = k, k \in \mathbb{R}$

<u>Proof</u>

We need to show that $f(x_1) = f(x_2), \forall x_1, x_2 \in (a,b)$

Let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$

Then, f = differentiable on $[x_1, x_2]$ (because $f'(x)$ exists $\forall x$)
f = continuous on $[x_1, x_2]$ (because any diff. fct. is cont.)

By the Mean Value Theorem $\Rightarrow$
there is $c \in (x_1, x_2)$ such that

$$\left.\begin{aligned} f'(c) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ \text{But } f'(x) &= 0 \quad \forall x \end{aligned}\right\} \Rightarrow f(x_2) - f(x_1) = 0$$

so $f(x_1) = f(x_2)$

Therefore, if $f'(x) = 0 \ \forall x$, $f(x) = k$, k = constant.

<u>Corollary 2</u>

Given $f'(x) = g'(x), \forall x \in (a,b)$

Prove There is k (constant) such that $f(x) = g(x) + k$ $\forall x \in (a,b)$

<u>Proof</u>

let $h(x) = f(x) - g(x)$

then $h'(x) = f'(x) - g'(x) = 0$

Then, by Corollary #1 $\Rightarrow h(x) = k$, k = constant

so $f(x) - g(x) = k$

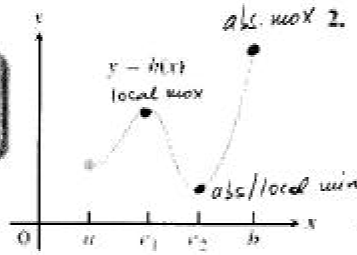
Section 4.1 – Exercises

1. (4.1 - #1 – 8)

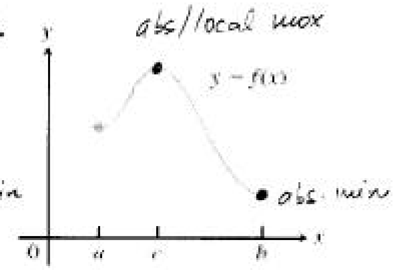
Find the extreme values and where they occur.

and local max. and min. points

1.



2.



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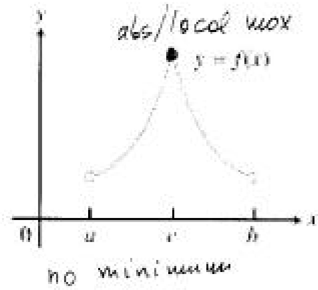

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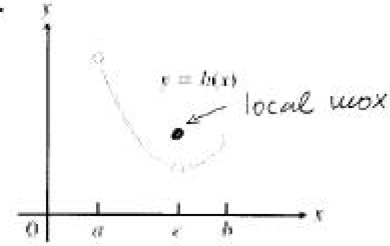

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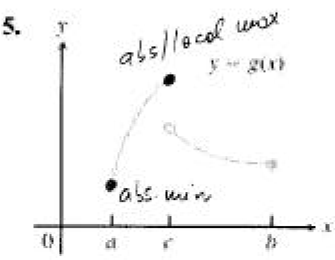

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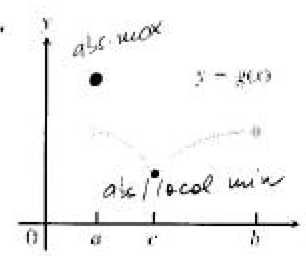
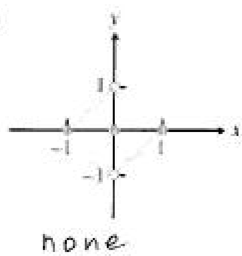
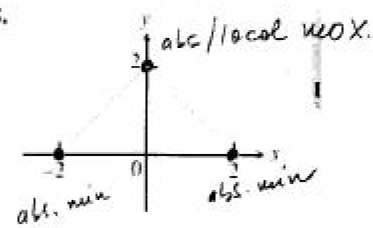


In Exercises 7–10, find the extreme values and where they occur.

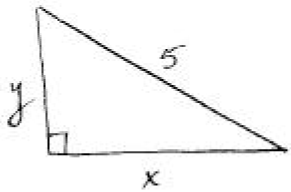
7.



8.



(2)



let x, y be the legs of the triangle

$x^2 + y^2 = 25 \Rightarrow y = \pm\sqrt{25 - x^2} \Rightarrow$

(y cannot be negative)

$y = \sqrt{25 - x^2}$

let $A(x)$ = area of the triangle in terms of x

$A(x) = \frac{1}{2} x \sqrt{25 - x^2}, \quad 0 \le x \le 5$

we want to find the absolute maximum of $A(x)$

• First, we find the critical points of $A(x)$

$A'(x) = \frac{1}{2} \cdot 1 \cdot \sqrt{25 - x^2} + \frac{1}{2} x \cdot \frac{1}{2\sqrt{25 - x^2}} (-2x)$

$A'(x) = \frac{1}{2} \left(\sqrt{25 - x^2} - \frac{x^2}{\sqrt{25 - x^2}} \right) = \frac{1}{2\sqrt{25 - x^2}} (25 - 2x^2)$

$A'(x) = 0$ iff $x^2 = \frac{25}{2}$ iff $x = \frac{5}{\sqrt{2}}$ (x cannot be negative)

$A'(x)$ not defined when $x = 5$ (x cannot be -5)

Therefore, the critical points are $x = \frac{5}{\sqrt{2}}$ and $x = 5$

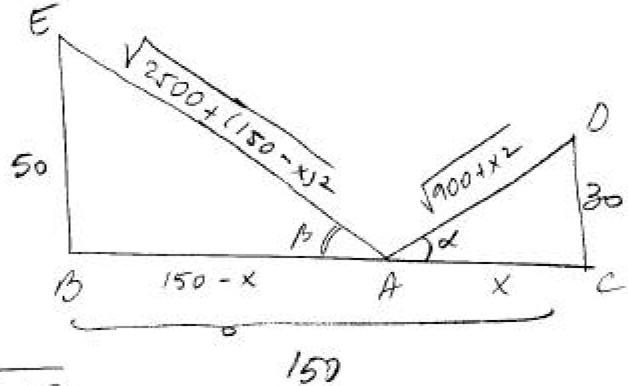
• Second, we evaluate $A(x)$ at the critical points and at the end points of the interval

$A\left(\frac{5}{\sqrt{2}}\right) = \frac{1}{2} \cdot \frac{5}{\sqrt{2}} \sqrt{25 - \frac{25}{2}} = \frac{5}{2\sqrt{2}} \cdot \frac{5}{\sqrt{2}} = \frac{25}{4}$

$A(5) = A(0) = 0$

Therefore, the largest positive area is $\frac{25}{4}$ cm^2 and it occurs when $x = \frac{5}{\sqrt{2}}$ cm and $y = \frac{5}{\sqrt{2}}$ cm

(3) Let $AC = x$
then $AB = 150 - x$
Let $L(x)$ = length of wire
$L(x) = EA + AD$



$$L(x) = \sqrt{50^2 + (150-x)^2} + \sqrt{30^2 + x^2}$$

$$L(x) = \sqrt{2500 + (150-x)^2} + \sqrt{900 + x^2}, \quad 0 \le x \le 150$$

we want to find the absolute minimum of $L(x)$

- First, we find the critical points of $L(x)$

$$L'(x) = \frac{1}{2\sqrt{2500+(150-x)^2}} \cdot 2(150-x)(-1) + \frac{1}{2\sqrt{900+x^2}}(2x)$$

$$L'(x) = \frac{x}{\sqrt{900+x^2}} - \frac{150-x}{\sqrt{2500+(150-x)^2}}$$

Note that if $\alpha = \angle CAD$ and $\beta = \angle BAE$, then

$L'(x) = \cos\alpha - \cos\beta$

$$\left.\begin{array}{l} L'(x) = 0 \text{ iff } \cos\alpha = \cos\beta \\ \text{But } \alpha, \beta \in (0°, 90°) \end{array}\right\} \Rightarrow \alpha = \beta$$

Also, $\triangle ACD \sim \triangle ABE$ $\Rightarrow$ $\dfrac{x}{30} = \dfrac{150-x}{50}$
$(\alpha = \beta$ and $\angle B = \angle C)$

$5x = 3(150 - x) \Rightarrow x = 56.25$ ft

The critical point is $x = 56.25$ ft.

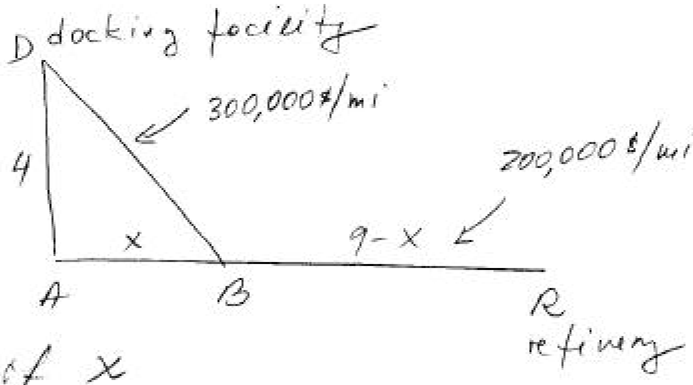
- Second, we evaluate $L(x)$ at the critical point and the endpoints

$L(0) = \sqrt{50^2 + 150^2} + \sqrt{900} \approx 188.11$

$L(150) = \sqrt{2500} + \sqrt{900 + 150^2} \approx 202.97$

$L(56.25) = \sqrt{2500 + (150 - 56.25)^2} + \sqrt{900 + (56.25)^2} \approx 160.13$

The min. length is 160.13 ft and it occurs when $x = 56.25$ ft



(7) Let $AB = x$
Then $BC = 9 - x$

Total cost = cost(BD) + cost(BR)

Let $C(x)$ = cost in terms of x

$$C(x) = 300{,}000\sqrt{16+x^2} + 200{,}000(9-x), \quad 0 \le x \le 9$$

we want to find the absolute min of $C(x)$

- First, we find the critical points of $C(x)$

$$C'(x) = 300{,}000 \cdot \frac{1}{2\sqrt{16+x^2}}(2x) + 200{,}000(-1)$$

$$C'(x) = \frac{300{,}000x}{\sqrt{16+x^2}} - 200{,}000$$

$$C'(x) = 0 \text{ iff } \frac{300{,}000x}{\sqrt{16+x^2}} = 200{,}000$$

iff
$$3x = 2\sqrt{16+x^2}$$
$$9x^2 = 4x^2 + 64$$
$$5x^2 = 64$$
$$x^2 = \frac{64}{5} \Rightarrow x = \frac{8}{\sqrt{5}} \approx 3.58 \text{ mi}$$
(x cannot be negative)

$C'(x)$ exists $\forall x$

The critical point is $x = 3.58$ mi

- Second, we evaluate $C(x)$ at the critical point and the end points

$$C(0) = 300{,}000 \cdot 4 + 200{,}000(9) = 3{,}000{,}000 \$$$
$$C(9) = 300{,}000\sqrt{16+81} = 2{,}954{,}660$$
$$C(3.58) = 300{,}000\sqrt{16+(3.58)^2} + 200{,}000(9-3.58) = 2{,}694{,}430$$

Therefore, the minimum cost is 2,694,430 million \$ and it occurs when $AB \approx 3.58$ mi