PART II Sections 2.2 & 2.3 - One-Sided Limits and Infinite Limits

Recall that in finding $\lim_{x \to a} f(x) = L$, f must be defined on both sides of a (not necessarily at a), and $f(x)$ must approach L as x approaches a from either side. That is why ordinary limits are called *two-sided*.

Example The Heaviside function H is defined by $H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$. This function is named after the electrical engineer Oliver Heaviside (1850 – 1925) and can be used to describe an electric current that is switched on at time $t = 0$.
What can you say about $\lim_{t \to 0} H(t)$?

Definition 1) We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a and less than a.

2) We write

$$\lim_{x \to a^+} f(x) = L$$

and say the **right-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the right**] is equal to L if we can make the values of $f(x)$ as close to L as we like by taking x to be sufficiently close to a and more than a.

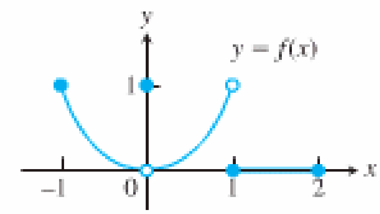
Theorem

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Exercise 1 Show that $\lim_{x\to 0}|x| = 0$.

Exercise 2 Prove that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Exercise 3 Find each limit (if it exists).



a) $\lim_{x\to -1^+} f(x)$ b) $\lim_{x\to 0^-} f(x)$ c) $\lim_{x\to 0^+} f(x)$

d) $\lim_{x\to 0} f(x)$ e) $\lim_{x\to 1} f(x)$ f) $\lim_{x\to 2^-} f(x)$

g) $\lim_{x\to 2^+} f(x)$

Exercise 4 Let $g(x)=\begin{cases} -x & \text{if } x\le -1 \\ 1-x^2 & \text{if } -1<x<1 \\ x-1 & \text{if } x>1 \end{cases}$.

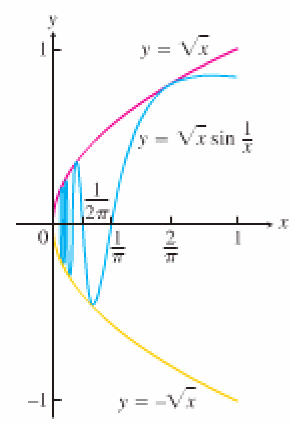
a) Evaluate each of the following limits, if it exists.

i) $\lim_{x\to 1^+} g(x)$ ii) $\lim_{x\to 1} g(x)$ iii) $\lim_{x\to 0} g(x)$

iv) $\lim_{x\to -1^-} g(x)$ v) $\lim_{x\to -1^+} g(x)$ vi) $\lim_{x\to -1} g(x)$

b) Sketch the graph of g.

Exercise 5 Let $g(x)=\sqrt{x}\sin\frac{1}{x},\ x>0$.



a) Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Exercise 6 Let $f(x)=\begin{cases} x & \text{if } -1\le x<0 \text{ or } 0<x\le 1 \\ 1 & \text{if } x=0 \\ 0 & \text{if } x>1 \text{ or } x<-1 \end{cases}$.

a) Graph the function.
b) Find the domain and range.
c) At what points c, if any, does $\lim_{x\to c} f(x)$ exist?
d) At what points does only the left-hand limit exist?
e) At what points does only the right-hand limit exist?

Exercise 7 Find the following limits:

a) $\lim_{x\to -2^+}\left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

b) $\lim_{x\to -2^-}(x+3)\frac{|x+2|}{x+2}$

c) $\lim_{x\to -2}\frac{2-|x|}{2+x}$

Exercise 8 Let $f(x)=\begin{cases} 3-x & if\ x<2 \\ \frac{x}{2}+1 & if\ x>2 \end{cases}$. Find the following:

a) $\lim_{x\to 2^+} f(x)$ b) $\lim_{x\to 2^-} f(x)$ c) $\lim_{x\to 2} f(x)$

d) $\lim_{x\to 4^+} f(x)$ d) $\lim_{x\to 4^-} f(x)$ e) $\lim_{x\to 4} f(x)$

Solutions

Important Limits

(1) $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

(2) $\lim_{t\to 0}\frac{\sin t}{t}=1$

Exercise 9 Find the following limits:

a) $\lim_{y\to 0}\frac{\sin 3y}{4y}$

b) $\lim_{t\to 0}\frac{2t}{\tan t}$

c) $\lim_{x\to 0}\frac{x+x\cos x}{\sin x\cos x}$

d) $\lim_{x\to 0}\frac{x\csc 2x}{\cos 5x}$

e) $\lim_{y\to 0}\frac{\sin 3y\cdot\cot 5y}{y\cot 4y}$

f) $\lim_{t\to 0}\frac{\sin(1-\cos t)}{1-\cos t}$

g) $\lim_{x\to 0}\frac{\sin 5x}{\sin 4x}$

h) $\lim_{x\to 0}\frac{\tan 3x}{\sin 8x}$

Answers: a) ¾; b) 2; c) 2; d) ½; e) 12/5; f) 1; g) 5/4; h) 3/8

Solutions

Infinite Limits (2.2)

Examples Sketch the graph of each function and use it to determine the limit when x approaches 0.

$f(x)=\frac{1}{x}$ $g(x)=\frac{1}{x^2}$

Definition 1 If f is a function defined on an open interval about a, except possibly at a itself, we write

$$\lim_{x\to a} f(x)=\infty$$

and say "the limit of $f(x)$, as x approaches a, is infinity"

if we can make the values of $f(x)$ arbitrarily large (as large as we like) by taking x sufficiently close to a (on either side of a) but not equal to a.

Note: This does not mean that ∞ is a number. Nor does it mean that the limit exists. It expresses the particular way in which the limit does not exist. It indicates that the values of $f(x)$ increase without bound as x becomes closer and closer to a.

Definition 2 If f is a function defined on an open interval about a, except possibly at a itself, we write

$$\lim_{x\to a} f(x)=-\infty$$

and say "the limit of $f(x)$, as x approaches a, is negative infinity"

if we can make the values of $f(x)$ arbitrarily large negative(as large as we like) by taking x sufficiently close to a (on either side of a) but not equal to a.

Note This indicates that the values of $f(x)$ decrease without bound as x becomes closer and closer to a.

Definition 3 The vertical line $x=a$ is called a **vertical asymptote** of the curve $y=f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x)=\infty \qquad \lim_{x\to a^-} f(x)=\infty \qquad \lim_{x\to a^+} f(x)=\infty$$

$$\lim_{x\to a} f(x)=-\infty \qquad \lim_{x\to a^-} f(x)=-\infty \qquad \lim_{x\to a^+} f(x)=-\infty$$

Exercise 10 Find the following limits:

a) $\lim_{x\to 3^+} \frac{2x}{x-3}$ b) $\lim_{x\to 1^-} \frac{1}{x^3-1}$ c) $\lim_{x\to 0^+} \ln x$

d) $\lim_{t\to\left(\frac{p}{2}\right)^-} \tan t$ e) $\lim_{x\to 0^+}\left(\frac{1}{x}-\ln x\right)$ f) $\lim_{x\to 0}\ln\left(\tan^2 x\right)$

g) $\lim_{x\to 0^-} \frac{5}{2x}$ h) $\lim_{x\to -8^+} \frac{2x}{x+8}$ i) $\lim_{x\to 0^-} \frac{2}{x^{\frac{1}{5}}}$

j) $\lim_{y\to 0} \frac{1}{y^{2/3}}$ k) $\lim_{t\to\left(-\frac{p}{2}\right)^+} \sec t$ l) $\lim_{x\to 0^-} (1+\csc x)$

m) $\lim_{x\to 1^+} \frac{x}{x^2-1}$ n) $\lim_{x\to -2^-} \frac{x^2-1}{2x+4}$ o) $\lim_{x\to 0^+}\left(\frac{1}{x^{1/3}}-\frac{1}{(x-1)^{4/3}}\right)$

Answers: a) ∞ ; b) $-\infty$; c) $-\infty$; d) ∞ ; e) ∞ ; f) $-\infty$; g) $-\infty$; h) $-\infty$; i) $-\infty$; j) ∞ ; k) ∞ ; l) $-\infty$; m) ∞ ; n) $-\infty$; o) ∞

Solutions