Sections 4.1 & 4.2

Some of the most important applications of differential calculus are optimization problems, in which we are required to find optimal (best) way of doing something.

These problems can be reduced to finding the maximum or minimum values of a function.

<u>Definition</u> Let f be a function with domain D. A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in D. The number $f(c)$ is called the **maximum value** of f on D.
Similarly, f has an **absolute minimum** at c if $f(c) \leq f(x)$ for all x in D. The number $f(c)$ is called the **minimum value** of f on D.

Note
- The maximum and minimum values of f are called the **extreme values** of f.

Examples

The Extreme Value Theorem (4.1)

This theorem gives conditions under which a function is guaranteed to have extreme values.

Hypothesis:	f continuous on a closed interval $[a,b]$
Conclusion:	f attains both an absolute maximum and an absolute minimum in $[a,b]$.

<u>Examples of functions that satisfy the hypothesis</u>:

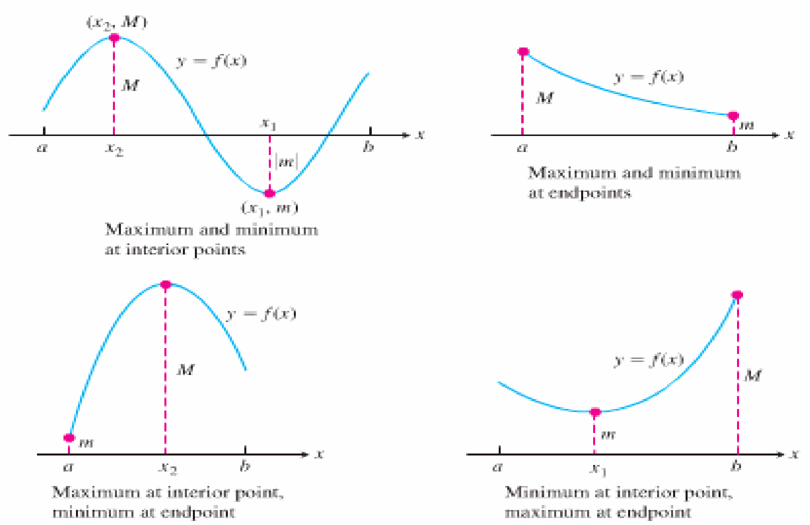


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

<u>Example of a function that does not satisfy the hypothesis:</u>

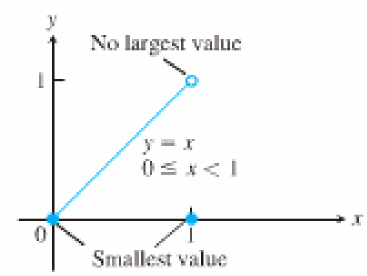


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

Notes:

- It is possible that an absolute minimum (or absolute maximum) to occur at more than one point in the interval.
- There are two requirements in The Extreme Value Theorem:
  - the interval be closed
  - the function be continuous.
- The Extreme Value Theorem says that a continuous function on a closed interval has an absolute maximum and a minimum value, but it does not tell us how to find these extreme values. We start by looking for local extreme values.

Definition A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c. (This means $f(c) \geq f(x)$ for all x in some open interval containing c.) The number $f(c)$ is called a **local maximum value** of f. Similarly, f has a **local minimum** (or **relative minimum**) at c if $f(c) \leq f(x)$ when x is near c. The number $f(c)$ is called a **local minimum value** of f.

Example

Fermat's Theorem (The First Derivative Theorem for Local Extreme Values) (4.1)

This theorem says that a function's derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined.

Hypothesis:	f is a function c is an interior point of the domain of f f has a local minimum or maximum value at c $f'(c)$ exists
Conclusion:	$f'(c) = 0$

Notes:

- Fermat's Theorem says that a function's derivative is always zero at an *interior point* where the function has a *local extreme value* and the *derivative is defined*.

- The Converse of the Fermat's Theorem is false in general:

  If $f'(c) = 0$, c is not necessarily a maximum or minimum .

  - Example

- There may be an extreme value where $f'(c)$ is not defined.

  - Example

- The only places where a function f can possibly have an extreme value (local or global) are:
  - interior points where $f' = 0$,
  - interior points where f' does not exist
  - endpoints of the domain of the function.

Definition A **critical number** of a function f is a number c in the domain of f such that either

$$f'(c) = 0 \text{ or } f'(c) \text{ does not exist.}$$

How to Find the Absolute Extrema of a Continuous Function *f* on a Closed Interval $[a, b]$

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Rolle's Theorem (4.2)

This theorem says that between any two points where a differentiable function crosses a horizontal line there is at least one point on the curve where the tangent is horizontal.

Hypothesis: f continuous on $[a, b]$

f differentiable on (a, b)

$f(a) = f(b)$

Conclusion: $\exists c \in (a, b)$ such that $f'(c) = 0$

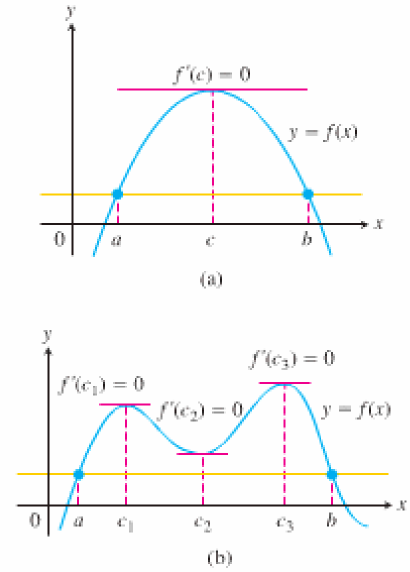


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

Corollary

Between any two zeros of a differentiable function on an interval, there is at least one zero of its derivative.

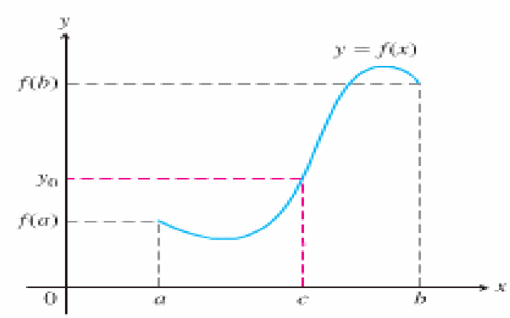
The Intermediate Value Theorem for Continuous Functions (2.5)

A function is said to have the Intermediate Value Property if whenever it takes on two values, it takes on all the values in between.

Hypothesis:	f continuous on a closed interval $[a,b]$
Conclusion:	f takes on every value between $f(a)$ and $f(b)$

Note

- The Intermediate Value Theorem is the reason the graph of a function continuous on an interval cannot have any breaks over the interval.

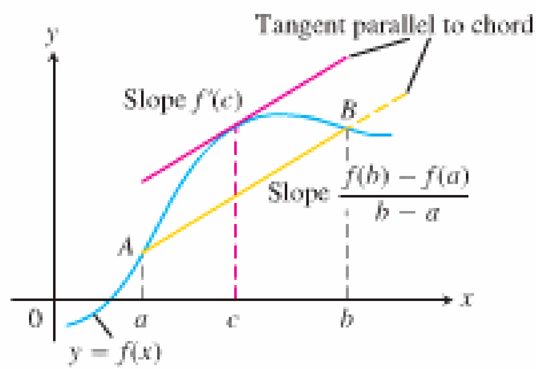


Corollary

If a function f is continuous, then any interval on which f changes sign contains a zero of the function.

The Mean Value Theorem (4.2)
This theorem says that if a function is differentiable, then there is a point somewhere between A and B where the tangent line is parallel to the secant line AB.

Hypothesis:	f continuous on a closed interval $[a,b]$ f differentiable on (a,b)
Conclusion:	$\exists c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$



Note

- The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from its derivative.

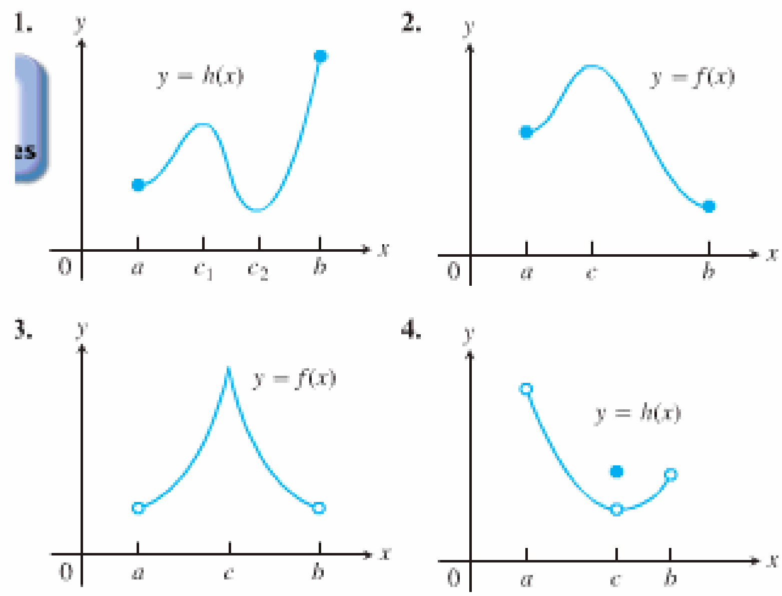
Corollaries of the Mean Value Theorem

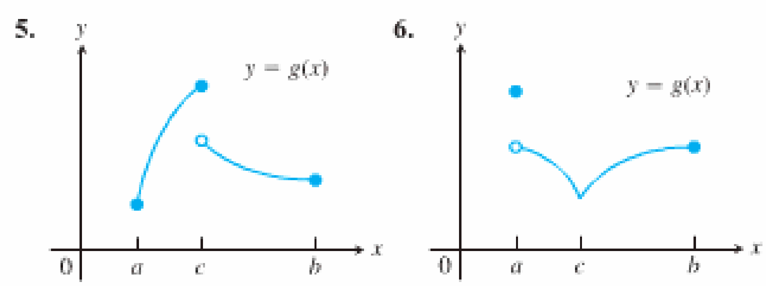
Corollary 1 Functions with Zero Derivatives Are Constant.

Corollary 2 Functions with the Same Derivative Differ by a Constant.

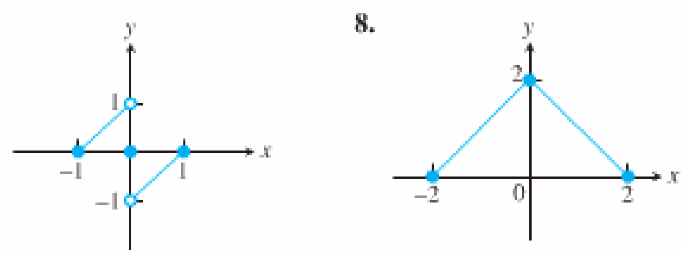
1.

Find the extreme values and where they occur.





In Exercises 7–10, find the extreme values and where they occur.



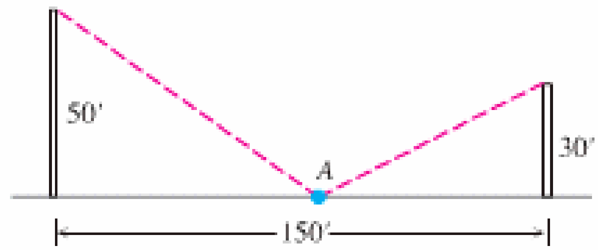
2.

What is the largest possible area for a right triangle whose hypotenuse is 5 cm long?

3.

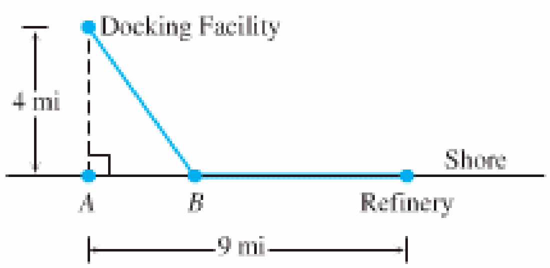
One tower is 50 ft high and another is 30 ft high. The towers are 150 ft apart. A guy wire is to run from point A to the top of each tower.

a) Locate point A so that the total length of guy wire is minimal.



4. Supertankers off-load oil at a docking facility 4 mi offshore. The nearest refinery is 9 mi east of the shore point nearest the docking facility. A pipeline must be constructed connecting the docking facility with the refinery. The pipeline costs $300,000 per mile if constructed underwater and $200,000 per mile if overla nd.

a) Locate point B to minimize the cost of the construction.



5. Find the critical numbers of each function:

a) $f(x) = x^3 + 6x^2 - 15x$ b) $f(x) = x^{\frac{3}{5}}(4 - x)$.

c) $g(t) = |3t - 4|$ $g(q) = 4q - \tan q$

6. Find the absolute maximum and minimum of each function on the given interval.

a) $f(x) = x^2$ on $[-2,1]$.

b) $g(x) = \sqrt{4 - x^2}$ on $[-2,1]$

c) $f(x) = x - 2\sin x$ on $[0, 2p]$

7. Find a function whose derivative is $\sin x$ and whose graph passes through $(0,2)$.

8. Show that $x^3 + 3x + 1 = 0$ has exactly one zero.

9. Find the velocity and displacement function of an object falling freely from rest with acceleration 9.8 m/s^2.

10. Find $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$ if $f(x) = x^2 + 2x - 1$ on $[0,1]$.

11. Find a, m, and b such that the given function is continuous on $[0,2]$ and differentiable on $(0,2)$.

$$f(x) = \begin{cases} 3, x = 0 \\ -x^2 + 3x + a, 0 < x < 1 \\ mx + b, 1 \le x \le 2 \end{cases}$$

12. Show that the equation $f(x) = x^4 + 4x + c = 0$ has at most two real roots.

13. If $f'(x) = 2x$ for any x, and $f(0) = 0$, find $f(2)$.

14. Find all possible functions with the given derivative.

a) $y' = \dfrac{1}{2\sqrt{x}}$ b) $y' = \dfrac{1}{\sqrt{x}}$ c) $y' = 4x - \dfrac{1}{\sqrt{x}}$

d) $y' = \sin 2t$ e) $y' = \cos\dfrac{t}{2}$

15. If $f'(x) = e^{2x}$, find $f(x)$ through $\left(0, \dfrac{3}{2}\right)$.

16. If $r'(t) = \sec t \tan t - 1$, find $r(t)$ through $(0,0)$.

17. If $v(t) = \dfrac{1}{t+2}, t > -2$ is the velocity function, find the position function $s(t)$ if $s(-1) = \dfrac{1}{2}$.

18. A trucker handed in a ticket at a toll booth showing that in 2 hours he had covered 159 mi on a toll road with a speed limit of 65 mph. The trucker was cited for speeding. Why?

<u>Answers:</u>

2) 25/4 sq. cm; 3) Min. length is 160.13 ft;4) Min. cost is 2,694,430 million dollars, when AB=3.58 mi; 5) 3/2, 0; 6c) abs. max is $\dfrac{5p}{3} + \sqrt{3}$, abs. min is $\dfrac{p}{3} - \sqrt{3}$; 10) ½; 11) a=3, m=1, b=4; 14) a) $\sqrt{x} + c$; b) $2\sqrt{x} + c$; c) $2x^2 - 2\sqrt{x} + c$ 16) $r(t) = \sec t - t - 1$; 17) $s(t) = \ln(t+2) + \dfrac{1}{2}$