Section 2.7 – Derivatives and Rates of Change – Part II
Section 2.8 – The Derivative as a Function

Derivatives

In the previous section we defined the slope of the tangent to a curve with equation $y = f(x)$ at the point $(a, f(a))$ to be

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We also saw that the velocity of an object with position function $s = f(t)$ at time $t = a$ is

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In fact, limits of the form $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

Definition 1 **The derivative of a function *f* at $x = a$,** denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (1) \qquad \text{or} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (2)$$

if this limit exists.

Notes:

- $f'(a)$ is the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$
- $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to *x* when $x = a$
- If we sketch the curve $y = f(x)$, when the derivative is large (and therefore the curve is steep), the *y*-values change rapidly. When the derivative is small, the curve is relatively flat, and the *y*-values change slowly.
- In particular, if $s = f(t)$ is the position of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement *s* with respect to the time *t*. In other words, $f'(a)$ is the *velocity of the particle at time* $t = a$. The *speed* of the particle is the absolute value of the velocity.

Exercise 1 Find the derivative of the function $f(t) = 2t^3 + t$ at the number a.

The Derivative as a Function

Now we change our point of view and let the number <u>a</u> vary. If we replace a in the equation (1) above we obtain

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (3)$$

For any number x for which this limits exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of** $\boldsymbol{f}$ and defined by equation (3). The value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent to the graph of f at the point $(x, f(x))$.

<u>Notes:</u>

- The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in equation (3).
- The domain of f' is the set $\{x | f'(x) \in \mathbb{R}\}$ and it may be smaller than the domain of f.
- If $f'(x)$ exists, we say that f **has a derivative at** $\boldsymbol{x}$ or **that** $\boldsymbol{f}$ **is differentiable at** $\boldsymbol{x}$**.**
- If f' exists at any x in the domain of f, we say that $\boldsymbol{f}$ **is differentiable.**
- The process of calculating a derivative is called **"differentiation**".

Exercise 2 Differentiate $f(x) = \dfrac{x}{x-1}$.

Exercise 3 a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
b) Illustrate by comparing the graphs of f and f'.

Exercise 4 If $f(x) = \sqrt{x+5}$, find the derivative of *f*. State the domain of f'.

Differentiable on an Interval; One-sided Derivatives

<u>Definition 2</u> 1) A function *f* is **differentiable on an open interval** (a,b) [or (a,∞) or $(-\infty,a)$ or $(-\infty,\infty)$] if and only if $f'(x)$ exists for any *x* in the interval.

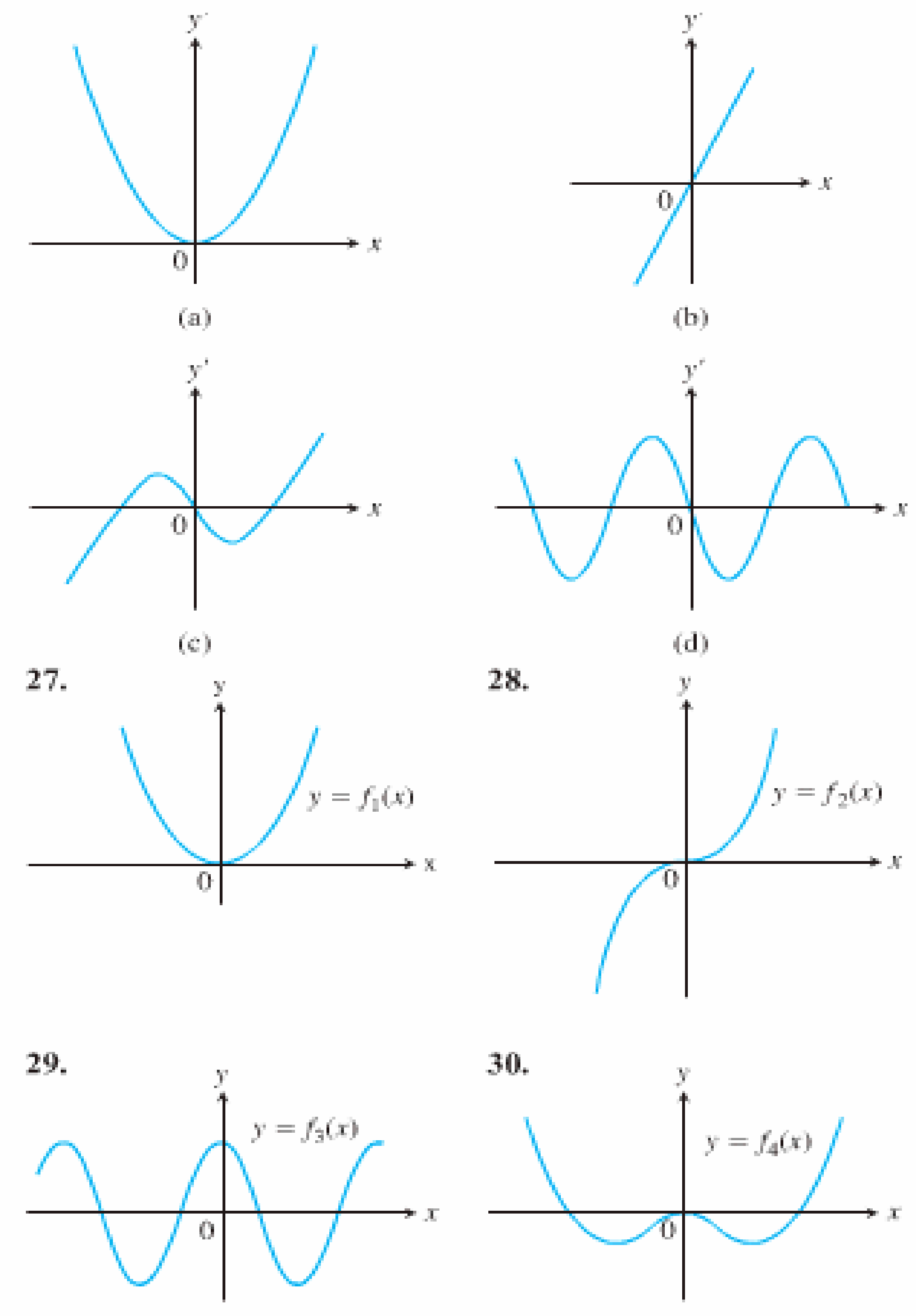
2) A function *f* is **differentiable on a closed interval** $[a,b]$ if and only if it is differentiable on (a,b) and if the right-hand derivative at *a* and the left-hand derivative at *b* exist.

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad f'_-(b) = \lim_{h\to 0^+} \frac{f(b+h) - f(b)}{h}$$

Exercise 5 Where is the function $f(x) = |x|$ differentiable? Give a formula for f'. Illustrate with the graphs of *f* and f'.

Exercise 6

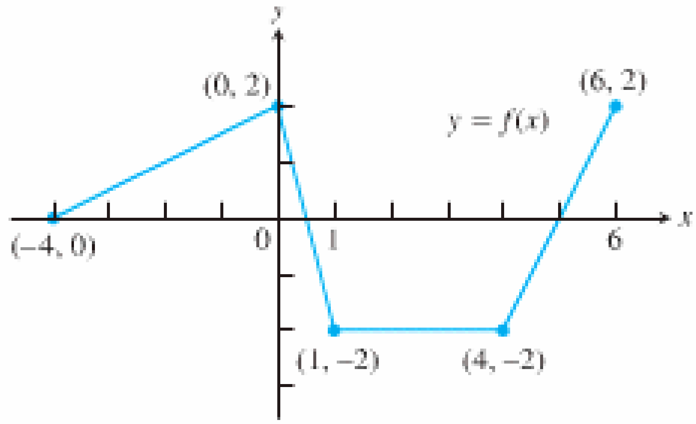
Match the functions graphed in #27 – 30 with the derivatives graphed in the accompanying figures a) – d).



Exercise 7

a) The graph of the accompanying figure is made of line segments joined end to end. At which points of the interval $[-4,6]$ is f' not defined? Give reasons for your answer.

b) Graph the derivative of f.



Exercise 8

Compare the right-hand and left-hand derivatives to show that the functions are not differentiable at the point P.

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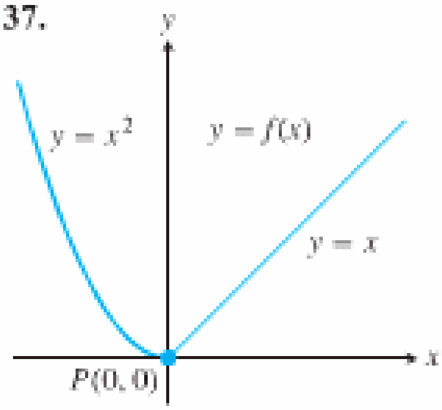

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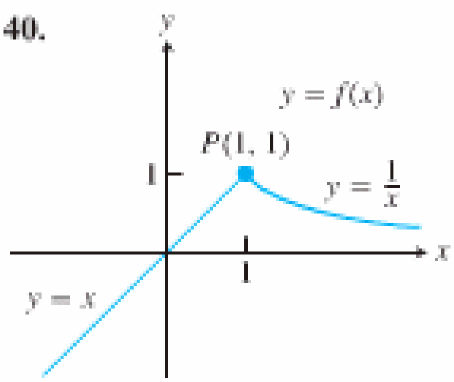

Other notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol dy/dx, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is a synonym for $f'(x)$.

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$\left.\frac{dy}{dx}\right|_{x=a}$ which is a synonym for $f'(a)$.

Theorem

If a function is differentiable at $x = a$, then f is continuous at $x = a$.

Proof

Notes 1) The converse is false: a function could be continuous, but not differentiable

2)The contrapositive is always true: if a function is not continuous at $x = a$,then it is not differentiable at $x = a$

In conclusion, **when does a function not have a derivative at a point?**

1) *a corner*, where the one-sided derivatives differ
2) *a cusp*, where the slope of the tangent approaches ∞ from one side and $-\infty$ from the other
3) *a vertical tangent*, where the slope of the tangent approaches ∞ from both sides or approaches $-\infty$ from both sides
4) *a discontinuity*

Higher Derivatives

If *f* is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of *f*.

Using Leibniz notation, we write $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$.

Exercise 9 Let $g(x)=1+\sqrt{4-x}$. Differentiate the function. Then find an equation of the tangent line at $(3,2)$.

Exercise 10 The figure shows the graph of a function over a closed interval. At what points does the function appear to be:

a) differentiable?
b) Continuous but not differentiable?
c) Neither continuous nor differentiable.

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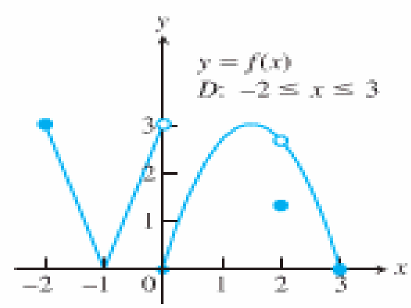

Exercise 11 Does any tangent to the curve $y=\sqrt{x}$ cross the *x*-axis at $x=-1$? If so, find an equation for the line and point of tangency. If not, why not?

Exercise 12 Show that $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$ is differentiable at $x=0$ and find $f'(0)$.

Exercise 13 a) If $g(x)=x^{\frac{2}{3}}$, show that $g'(0)$ does not exist.

b) If $a\neq 0$, find $g'(a)$.

c) Show that the graph has a vertical tangent line at $(0,0)$

Answers:1) $6a^2+1$; 2) $\frac{-1}{(x-1)^2}$; 3) $3x^2-1$; 4) $\frac{1}{2\sqrt{x+5}}$; 5) all x except 0; 6) 27 – b, 28 – a, 29 – d, 30 – c; 9b) $x+2y=7$; 11) $y-1=\frac{1}{2}(x-1)$; 12) $f'(0)=0$.

Solutions