PART I Sections 2.2 & 2.3 – Limit of a Function and Limit Laws

In section 2.1 we saw how limits arise when we want to find the tangent to a curve or the velocity of an object. Now we turn our attention to limits in general and methods for computing them. We will begin with an informal definition of limit .

Example 1 Let's investigate the behavior of the function ________________________ for values of x near 2.

Similarly, we define the notion of a limit for an arbitrary function.

Definition 1 If f is a function defined on an open interval about a, except possibly at a itself, we write

$$\lim_{x\to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ become closer and closer to the number L as x approaches the number a (from either side of a) but $x \neq a$.

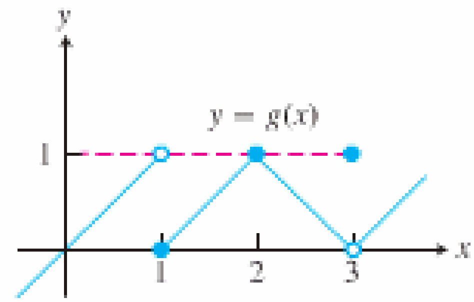
Notes: - In finding the limit of $f(x)$ as $x \to a$ we never consider $x = a$.
- $f(x)$ need not even be defined when $x = a$
- The only thing that matters is how f is defined *near* a

Example 2 Here we have three functions. This example illustrates that the limit of a function does not depend on how the function is defined at the point being approached.

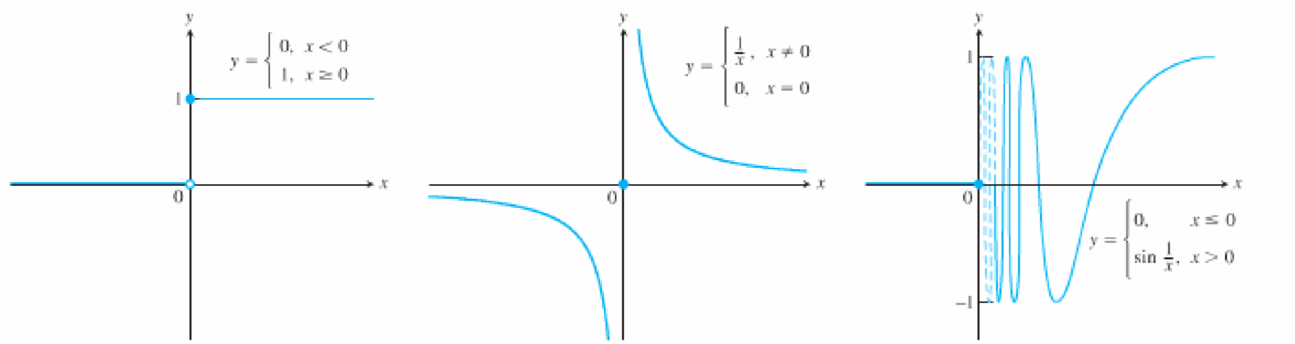
Exercise 1 For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

a) $\lim_{x\to 1} g(x)$ b) $\lim_{x\to 2} g(x)$

c) $\lim_{x\to 3} g(x)$ d) $\lim_{x\to 2.5} g(x)$



Example3 Here are some ways that limits can fail to exist.



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

Theorem – Limit Laws (2.3)

This theorem shows us how to calculate limits of functions that are arithmetic combinations of functions having known limits.

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

1. Sum Rule: $\lim_{x\to a}\left(f(x)\pm g(x)\right)=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$

2. Product Rule: $\lim_{x\to a}\left(f(x)\cdot g(x)\right)=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

3. Quotient Rule: $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},\lim_{x\to a} g(x)\neq 0$

4. Constant Multiple Rule: $\lim_{x\to a}\left(c\cdot f(x)\right)=c\cdot\lim_{x\to a} f(x), c\in\mathbb{R}$

5. Power Rule: $\lim_{x\to a}\left[f(x)\right]^n=\left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer

6. Root Rule: $\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$, where n is a positive integer.

If n is even, we assume that $\lim_{x\to a} f(x)>0$

Special Limits (2.3)

The Constant Function $\lim_{x \to a} c = c$

The Identity Function $\lim_{x \to a} x = a$

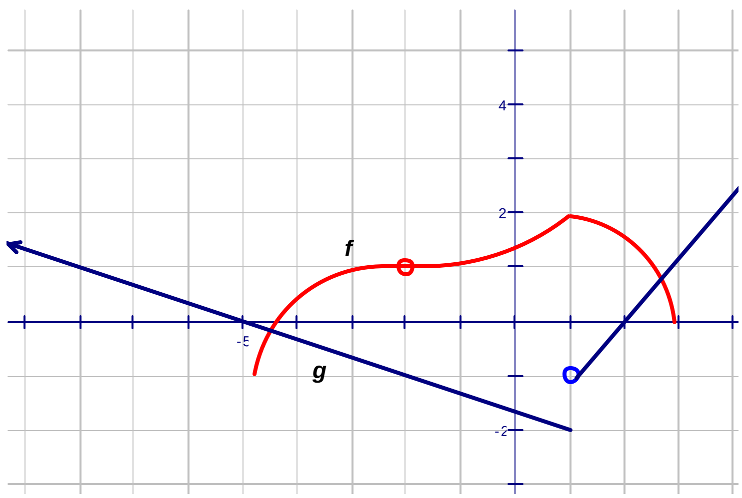
Direct Substitution Property If $f(x)$ is a polynomial or a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$

Not all limits can be evaluated by direct substitution.

Note: **Special Cases** – these are cases where the Limit Laws cannot be applied.

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 1^{\infty}, 0^0, \infty^0$$

Exercise 2 Use the Limit Laws and the graphs of f and g to evaluate the following limits, if they exist:



a) $\lim_{x \to -2} \left[f(x) + 5g(x) \right]$

b) $\lim_{x \to 1} \left[f(x) g(x) \right]$

c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

Exercise 3 Evaluate the following limits:

a) $\lim_{x \to 5} \left(2x^2 - 3x + 4 \right)$

b) $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

c) $\lim_{x \to 0} \left(2\sin x - 1 \right)$

d) $\lim_{x \to 0} \sec x$

e) $\lim_{x \to 2} \left(-x^2 + 5x - 2 \right)$

f) $\lim_{z \to 0} \left(2z - 8 \right)^{\frac{1}{3}}$

g) $\lim_{t \to 0} \left(t^3 + \frac{\cos 5t}{10,000} \right)$

Solutions

Note: Functions with the Direct Substitution Property are called *continuous at a* and will be studied in Section 2.5. However, not all limits can be evaluated by direct substitution.

Eliminating Zero Denominators Algebraically (2.3)

Some of the recommended methods are : simplifying the rational expression, rationalizing the numerator or denominator, substitution.

Exercise 4 Find the following limits:

a) $\lim_{x\to 1}\frac{x^2-1}{x-1}$ b) $\lim_{x\to -3}\frac{x+3}{x^2+4x+3}$ c) $\lim_{y\to 0}\frac{5y^3+8y^2}{3y^4-16y^2}$

d) $\lim_{u\to 1}\frac{u^4-1}{u^3-1}$ e) $\lim_{x\to 1}\frac{x^2-1}{x^2+x-2}$ f) $\lim_{t\to 0}\frac{t^2+t-2}{t^2-1}$

g) $\lim_{t\to 1}\frac{t^2+t-2}{t^2-1}$ h) $\lim_{h\to 0}\frac{(2+h)^3-8}{h}$ i) $\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}$

j) $\lim_{x\to 4}\frac{4x-x^2}{2-\sqrt{x}}$ k) $\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}$ l) $\lim_{h\to 1}\frac{\sqrt{5h+4}-2}{h}$

m) $\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}$ n) $\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}$ o) $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$

p) $\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$ r) $\lim_{x\to 0}\frac{\sqrt[3]{1+cx}-1}{x}$ s) $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$

Answers: a) 2; b) -1/2; c) -1/2; d) 4/3; e) 2/3; f) 2; g) 3/2; h) 12; i) 2/3; j) 16; k) -1/3; l) 1; m) 5/4; n) -1; o) 1; p) 1/2; r) c/3; s) 2/3

Exercise 5 Evaluate $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for the given value of *x* and function *f*.

a) $f(x)=x^2, x=-2$ b) $f(x)=\sqrt{x}, x=7$

Answers: a) -4; b) $\frac{1}{2\sqrt{7}}$.

Exercise 6 Find $\lim_{x\to 1} g(x)$ if $g(x)=\begin{cases} x+1 & \text{if } x\neq 1 \\ p & \text{if } x=1 \end{cases}$.

Solutions

The next two theorems give two additional properties of limits.

The Squeeze Theorem (The Sandwich Theorem) (2.3)

Hypothesis: (1) $g(x) \le f(x) \le h(x)$ for all x in some open interval containing a , except possibly at a *and* (2) $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = L$ Conclusion: $\lim\limits_{x\to a} f(x) = L$

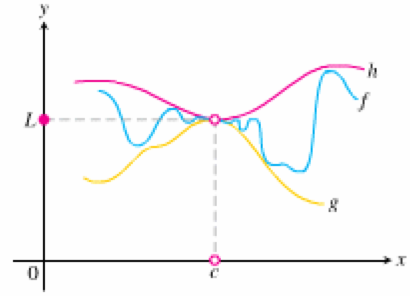


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

Theorem

Hypothesis:	(1) $f(x) \le g(x)$ for all x in some open interval containing a , except possibly at a *and* (2) $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist
Conclusion:	$\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$

Exercise 7 If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim\limits_{x\to 0} f(x)$.

Exercise 8 Find the following limits:

a) $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x}$ b) $\lim\limits_{x\to 0} x^2 \cos(20px)$ c) $\lim\limits_{x\to 0} x \sin\frac{1}{x}$ d) $\lim\limits_{x\to 0} \sqrt{x^3+x^2}\sin\frac{p}{x}$

Exercise 9 If $x^4 \le f(x) \le x^2$ for $-1 \le x \le 1$, and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limit at these points?

Exercise 10 Is there a number a such that $\lim\limits_{x\to -2} \frac{3x^2+ax+a+3}{x^2+x-2}$ exist? Of so, find the value of a and the value of the limit.

Exercise 11 Evaluate $\lim\limits_{x\to 0} \frac{|2x-1|-|2x+1|}{x}$.

Solutions