Section 5.3
The Definite Integral

We saw in Section 5.1 that when attempting to estimate the area under a curve or the distance traveled by an object , we arrived at limits of the form

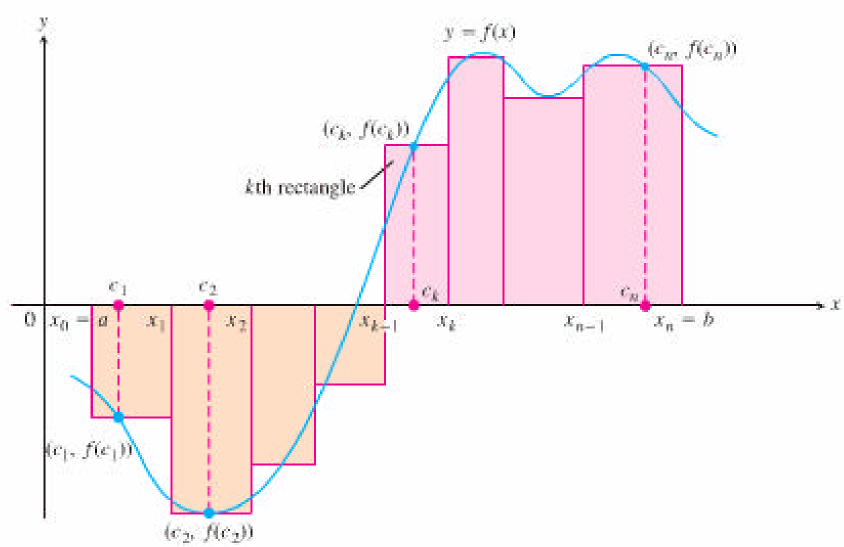
$$(1) \qquad \lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k\right)\Delta x = \lim_{n\to\infty}\left[\, f\left(x_1\right)\Delta x + f\left(x_2\right)\Delta x + \ldots + f\left(x_n\right)\Delta x\right]$$

where *f* was a nonnegative continuous function defined on a closed interval $\left[a,b\right]$, $x_1, x_2, \ldots x_n$ where points in $\left[a,b\right]$, and $\Delta x = \dfrac{b-a}{n}$ was the width of each subinterval $\left[x_{k-1}, x_k\right]$.

The same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. We will see that limits of the form (1) also arise in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as other quantities. We give this type of limit a special name and notation.

Definition of a Definite Integral If *f* is a continuous function defined on $\left[a,b\right]$, we divide the interval $\left[a,b\right]$ into *n* subintervals of equal width $\Delta x = \dfrac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots x_n = b$ be the endpoints of these subintervals and we choose sample points $c_1, c_2, \ldots c_n$ in these subintervals, so c_k lies in the kth subinterval $\left[x_{k-1}, x_k\right]$.
Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f\left(x\right)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(c_k\right)\Delta x$$



- Note 1 $\int_a^b f(x)\,dx$
  - The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums.
  - $f(x)$ is called the **integrand**.
  - a and **b** are called the **limits of integration**; a is the **lower limit** and b is the **upper limit.**
  - The symbol dx has no official meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol.
  - The procedure of calculating an integral is called **integration**.

- Note 2 The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. We could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

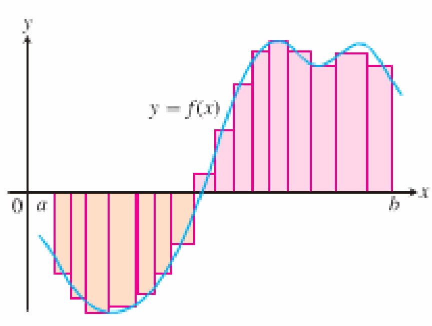
- Note 3 Because we have assumed that *f* is continuous, the limit in the definition always exists and gives the same value no matter how we choose the sample points c_k.
  The limit in the definition also exists if f has a finite number of removable or ju mp discontinuities (but not infinite discontinuities). We can also define the definite integral for such functions.

- Note 4 The sum $\sum_{k=1}^{n} f(c_k)\,\Delta x$ is called a **Riemann sum**.
  The set $\{x_0, x_1, x_2, \ldots x_n\}$ is called a **partition of** $[a,b]$**.**

- Note 5 If *f* takes on both positive and negative values, as in the above figure, then the Riemann sum is the sum of the areas of the rectangles that lie above the *x*-axis and the *negatives* of the areas of the rectangles that lie below the *x*-axis. When we take the limit of such Riemann sums, we get the situation illustrated below. A definite integral can be interpreted as a *net area*, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the *x*-axis and below the graph of *f* and A_2 is the area of the region below the *x*-axis and above the graph of *f*.



- Note 6 Although we have defined $\int_a^b f(x)\,dx$ by dividing $[a,b]$ into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width. If the subintervals have different widths, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width approaches 0.

Expressing Limits as Integrals

Exercise 1 Express $\lim_{n\to\infty}\sum_{k=1}^{n}\left[c_k^{\,3}+c_k\sin c_k\right]\Delta x$ as an integral on the interval $[0,p]$.

- Note: When we apply the definite integral to physical situations, it will be important to recognize limits of sums as integrals. In general when we write

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x=\int_a^b f(x)\,dx$$

we replace $\lim\sum$ by $\int$, c_k by x, and Δx by dx.

Evaluating Integrals

When we use the definition to evaluate a definite integral, we need to know how to work with sums.

$$\sum_{k=1}^{n} k=\frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} c = nc$$

$$\sum_{k=1}^{n} k^2=\frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} k^3=\left[\frac{n(n+1)}{2}\right]^2 \qquad \sum_{k=1}^{n}(a_k \pm b_k)=\sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

Exercise 2 a) Evaluate the Riemann sum for $f(x)=x^3-6x$ taking the sample points to be right-hand endpoints and $a=0, b=3, n=6$.

b) Evaluate $\int_0^3\left(x^3-6x\right)dx$.

Using Area to Evaluate Definite Integrals

Exercise 3 Evaluate the following integrals by interpreting each in terms of areas.

a) $\int_0^1 \sqrt{1-x^2}\,dx$

b) $\int_0^3 (x-1)\,dx$

c) $\int_{-2}^1 |x|\,dx$

Properties of the Definite Integral

1. Order of Integration $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. Zero Width Interval $\int_a^a f(x)\,dx = 0$

3. Constant Function $\int_a^b c\,dx = c(b-a)$

4. Constant Multiple $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

5. Sum and Difference $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

6. Additivity $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

7. Max – Min Inequality If $m \le f(x) \le M$ for $x \in [a,b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

8. Domination If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$

If $f(x) \ge g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Exercise 4 a) Evaluate $\int_0^1 (4+3x^2)\,dx$

b) If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$.

c) Use Property 7 to estimate $\int_0^1 e^{-x^2}\,dx$