Section 5.1 – Estimating with Finite Sums

In this section we discover that in attempting to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.

The Area Problem

In trying to solve this problem we have to ask: What is the meaning of the word *area*? This question is easy to answer for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles and adding the areas of the triangles.

However, it is not easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

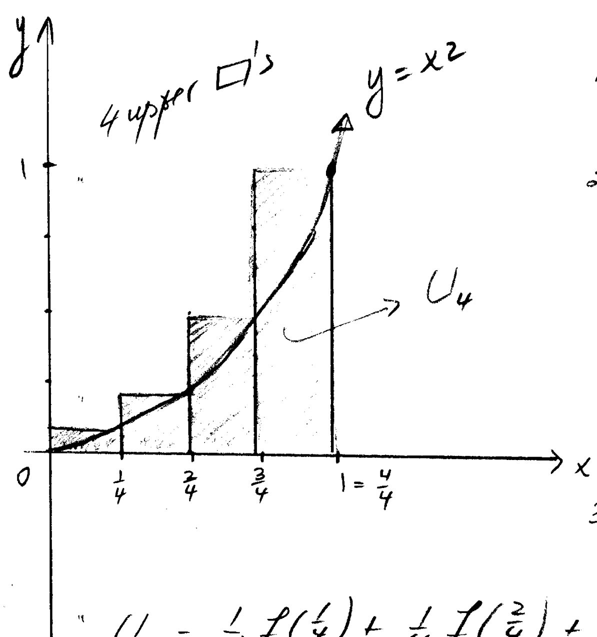
Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant line and then took the limit of these approximations. We pursue a familiar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

The following example illustrates the procedure.

Example 1
a) Use four upper rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.
b) Repeat a) using lower rectangles.
c) Write a formula for the sum of the upper n rectangles and show that it approaches $\frac{1}{3}$.
d) Repeat d) for lower rectangles.

Let A - the area under $y = x^2$
Note that $0 < A < 1$

(a)



1st - divide $[0,1]$ into 4 equal parts (subintervals)

2nd - draw the upper rectangles
The base is the same for all $= \frac{1}{4}$
The height is the value of f at the right-hand endpoint of the subinterval

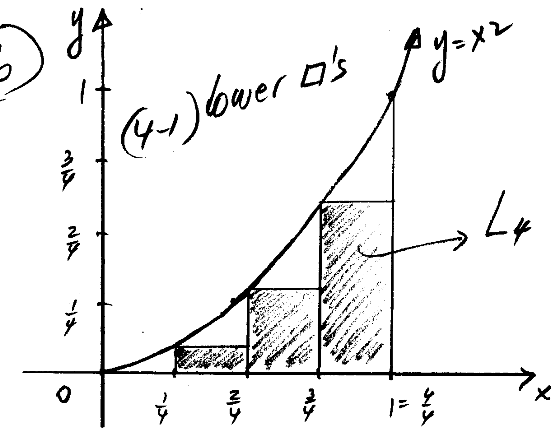
3rd - let U_4 = sum of the areas of the upper rectangles

$$U_4 = \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f\left(\frac{4}{4}\right)$$

$$U_4 = \frac{1}{4} \sum_{k=1}^{4} f\left(\frac{k}{4}\right) = \frac{1}{4} \sum_{k=1}^{4} \left(\frac{k}{4}\right)^2 = \frac{1}{4} \cdot \frac{1}{16} \sum_{k=1}^{4} k^2 = \frac{1}{4} \cdot \frac{1}{16} \cdot \frac{4(4+1)(2 \cdot 4+1)}{6}$$

$$U_4 = \frac{15}{32} \approx 0.46875$$ Therefore, $A < 0.46875$

(b)



1st - divide [0,1] into 4 equal parts (subintervals)

2nd - draw the lower rectangles
The base is the same for all $= \frac{1}{4}$
The height is the value of f at the left-hand endpoint of the subinterval.

3rd - Let L_4 = sum of the areas of the lower rectangles

$$L_4 = \frac{1}{4}f(0) + \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{2}{4}\right) + \frac{1}{4}f\left(\frac{3}{4}\right)$$

$$L_4 = \frac{1}{4}\sum_{k=0}^{3} f\left(\frac{k}{4}\right) = \frac{1}{4}\sum_{k=0}^{3}\left(\frac{k}{4}\right)^2 = \frac{1}{4}\cdot\frac{1}{16}\sum_{k=0}^{3} k^2 = \frac{1}{4}\cdot\frac{1}{16}\cdot\frac{3(3+1)(2\cdot 3+1)}{6}$$

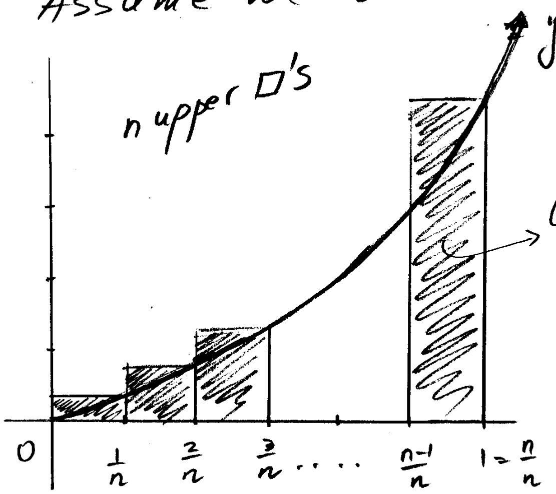
$$L_4 = \frac{7}{32} \approx 0.21875$$

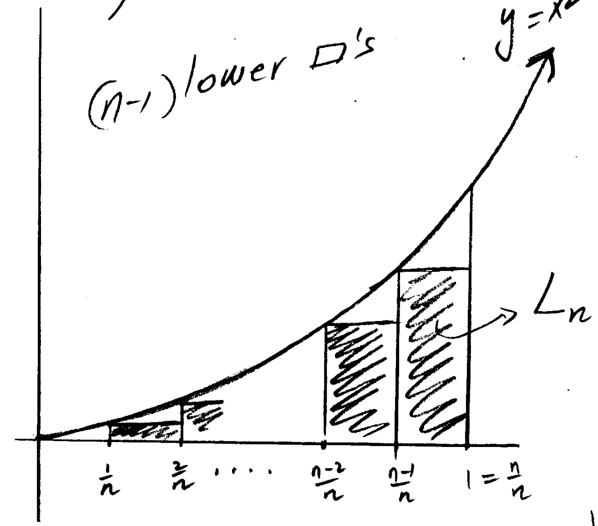
Therefore, $\boxed{0.21875 < A < 0.46875}$

<u>Note</u>: We can repeat the process with a larger number of strips (rectangles). The larger the number of strips, the better the estimates for A are.

(c) and (d)

Assume we divide $[0,1]$ into n equal subintervals





$$U_n = \sum_{k=1}^{n} U_k = \sum_{k=1}^{n} \frac{1}{n}\cdot f\left(\frac{k}{n}\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2} = \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$U_n = \frac{(n+1)(2n+1)}{6n^2}$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty}\left(\frac{1}{6}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)$$

$$= \frac{1}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{6}\cdot 1\cdot 2 = \frac{1}{3}$$

$$\lim_{n\to\infty} U_n = \frac{1}{3}$$

$$L_n = \sum_{k=0}^{n-1} U_k = \sum_{k=0}^{n-1}\frac{1}{n} f\left(\frac{k}{n}\right)$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\frac{k^2}{n^2} = \frac{1}{n^3}\sum_{k=0}^{n-1} k^2$$

$$= \frac{1}{n^3}\,\frac{(n-1)(n-1+1)(2(n-1)+1)}{6}$$

$$L_n = \frac{(n-1)\,n(2n-1)}{6n^3} = \frac{(n-1)(2n-1)}{6n^2}$$

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2}$$

$$= \frac{1}{6}\lim_{n\to\infty}\left(\frac{n-1}{n}\cdot\frac{2n-1}{n}\right)$$

$$= \frac{1}{6}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= \frac{1}{6}\cdot 1\cdot 2$$

$$\lim_{n\to\infty} L_n = \frac{1}{3}$$

but $\left.\begin{array}{l} L_n < A < U_n \\ \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n \end{array}\right\}$ $\Longrightarrow$ We define the area A to be the limit of the sums of the areas of the approximating □'s, that is

$$\boxed{A = \lim_{\infty} L_n = \lim_{\infty} U_n = \frac{1}{3}}$$

The Distance Problem

Let's consider the distance problem: Find the distance traveled by an object during a certain time period if the velocity of the object is know at all times. If the velocity remains constant, then the distance problem is easy to solve by means of the formula

$$\text{Distance} = \text{velocity x time}$$

But if the velocity varies, it is not so easy to find the distance traveled.

<u>Example 2</u> Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table (in order to have the time and the velocity in consistent units, we first convert the velocity readings from mph to feet per second 1 mi/h = 5280/ 3600 ft/s):

Time (s)	0	5	10	15	20	25
Velocity (ft/sec)	25	31	35	43	47	46

For each time interval we assume that the velocity doesn't change much, so we can estimate the distance traveled during that time by assuming that the velocity is constant.

Estimate the total distance traveled by the car.

a) Use the velocity at the beginning of each time period as the assumed constant velocity.
b) Repeat the problem using the velocity at the end of each time period instead of the velocity at the beginning as the assumed constant velocity .
c) Sketch a graph of the velocity function of the car.

(a) Use the velocity at the beginning of each time interval and approximate the distance traveled:

from $t=0s$ to $t=5s$ $\quad 5s \cdot 25\,ft/s = 125\,ft$

from $t=5s$ to $t=10s$ $\quad 5s \cdot 31\,ft/s = 155\,ft$

from $t=10s$ to $t=15s$ $\quad 5s \cdot 35\,ft/s = 175\,ft$

from $t=15s$ to $t=20s$ $\quad 5s \cdot 43\,ft/s = 215\,ft$

from $t=20s$ to $t=25s$ $\quad 5s \cdot 47\,ft/s = 235\,ft$

The estimate for the total distance traveled is:

$5 \cdot 25 + 5 \cdot 31 + 5 \cdot 35 + 5 \cdot 43 + 5 \cdot 47 =$

$5(25+31+35+43+47) = 905\,ft$

(b) Use the velocity at the end of each time interval to estimate the distance traveled.

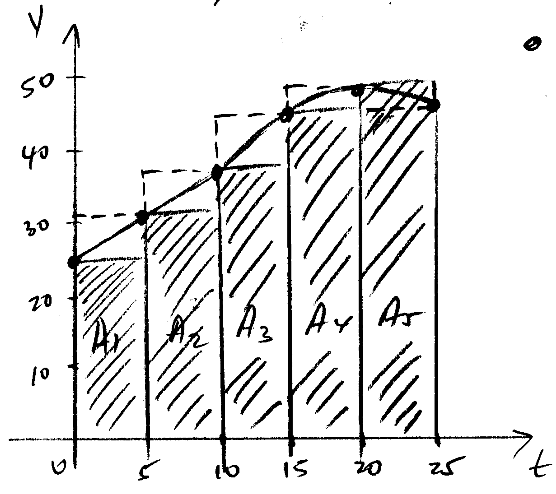
$5(31) + 5(35) + 5(43) + 5(47) + 4(46) =$

$5(31 + 35 + 43 + 47 + 46) = 1010$ ft

Therefore, the distance traveled d

$\boxed{905 \text{ ft} < d < 1010 \text{ ft}}$

Note: These calculations remind us of the sums we used earlier to estimate area. The similarity is explained when we sketch a graph of the velocity function of the car.



- Draw the rectangles whose heights are the initial velocity for each time interval

$A_1 = 5 \cdot 25 \approx$ distance traveled in the first 5 seconds

$A_2 = 5 \cdot 31 \approx$ dist. traveled from $t=5$ to $t=10$ s

$A_3 = 5 \cdot 35$

$A_4 = 5 \cdot 43$

$A_5 = 5 \cdot 47$

The sum of the areas of the rectangles is $L_5 = 905$ our estimate for the distance traveled.

- Draw □'s whose heights are the velocity at the end of each time interval and got $U_5 = 1010$

The exact distance traveled is d

$d = \lim_{\infty} L_n = \lim_{\infty} U_n =$ area under the curve of the velocity function