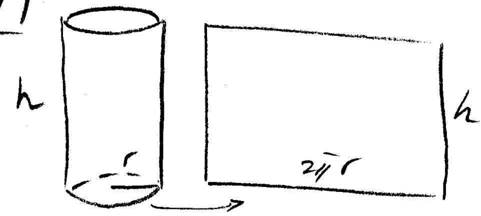
HANDOUT 4.5 - SOLUTIONS

PROBLEM 1



let r = radius of base (cm)
h = height of cilinder (cm)

$$\left.\begin{array}{l} V = \pi r^2 h \ \text{cm}^3 \\ V = 1 \text{ liter} = 1000 \text{ cm}^3 \end{array}\right\} \Rightarrow \pi r^2 h = 1000 \quad (*)$$

least material iff total area is minimum

let A = total area

$$\left.\begin{array}{l} A = 2(\pi r^2) + 2\pi r h \\ (*)\ \pi r^2 h = 1000 \Rightarrow h = \dfrac{1000}{\pi r^2} \end{array}\right\} \Rightarrow$$

$$A = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$$

$$A = 2\pi r^2 + \frac{2000}{r}, \text{ with } r > 0$$

$$\frac{dA}{dr} = 4\pi r + 2000 \cdot \frac{-1}{r^2}$$

$$\frac{dA}{dr} = A' = 4\pi r - \frac{2000}{r^2}$$

$$A' = 0 \Rightarrow \frac{4\pi r^3 - 2000}{r^2} = 0$$

$$4\pi r^3 = 2000$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \quad \text{- critical point}$$

Find if $r = \sqrt[3]{\frac{500}{\pi}}$ is a min or max.

$$A'' = 4\pi - 2000(-2)r^{-3}$$

$$A'' = 4\pi + \frac{4000}{r^3}$$

$$A''\left(\sqrt[3]{\frac{500}{\pi}}\right) = 4\pi + \frac{4000\pi}{500} > 0$$

therefore $r = \sqrt[3]{\frac{500}{\pi}}$ is a minimum

So, we use the least material if $r = \sqrt[3]{\frac{500}{\pi}}$ and $h = \frac{1000}{\pi r^2}$

$r \approx 5.42$ cm $\quad h \approx 10.84$ cm

PROBLEM 2 $r(x) = 9x$ revenue
$c(x) = x^3 - 6x^2 + 15x$ cost

let $p(x)$ = profit
$p(x) = r(x) - c(x)$
$p(x) = -x^3 + 6x^2 - 6x$

Find critical points:

$$p'(x) = -3x^2 + 12x - 6$$

$$p'(x) = 0 \Rightarrow -3(x^2 - 4x + 2) = 0$$

$$x = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

Find if $x_1 = 2-\sqrt{2}$ and $x_2 = 2+\sqrt{2}$ are min. or max.

$$p''(x) = -6x + 12$$

$$p''(2-\sqrt{2}) = -6(2-\sqrt{2}) + 12 = -12 + 6\sqrt{2} + 12 > 0$$

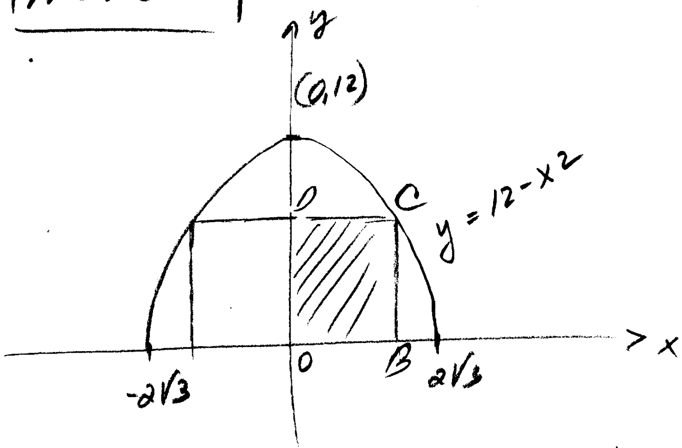
so $x_1 = 2-\sqrt{2}$ is a minimum

$$p''(2+\sqrt{2}) = -6(2+\sqrt{2}) + 12 = -12 - 6\sqrt{2} + 12 < 0$$

so $x_2 = 2+\sqrt{2}$ is a maximum

& max. profit occurs at $x = 2+\sqrt{2}$
$x \approx 3.414$ thousand and $P_{max} = p(2+\sqrt{2}) \approx 9.657$

PROBLEM 3



$y=12-x^2$ is a parabola opening downward with x-int: $12-x^2=0$, $x=\pm 2\sqrt{3}$
and vertex $(0,12)$

Area rectangle is max iff Area (OABC) is max.

let $B(x,0)$
then $C(x, 12-x^2)$
Let $A(x)$ = area of OBCD
$A(x) = x(12-x^2) = 12x - x^3$
$0 \le x \le 2\sqrt{3}$

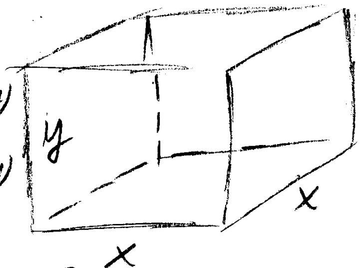
Find critical points:
$A'(x) = 12 - 3x^2$
$A'(x) = 0 \Rightarrow 3(4-x^2) = 0$
$x = 2$ or $\cancel{x=-2}$ (not in I)

Evaluate $A(x)$ at endpoints and critical point:
$A(0) = 0$
$A(2) = 2(12-4) = 32$
$A(2\sqrt{3}) = 0$

Therefore, the maximum area occurs when $x = 2$; the dimensions of the rectangle must be 4 units by $12-2^2 = 8$ units

PROBLEM 4

$V = 500\ ft^3$



let x = length of the base (ft)
let y = height (ft)

$V = x^2 y = 800$ (*)

We minimize the weight iff we minimize the surface area S

$S = x^2 + 4xy$
(*) $x^2y = 500 \Rightarrow y = \frac{500}{x^2}$
$\Rightarrow$

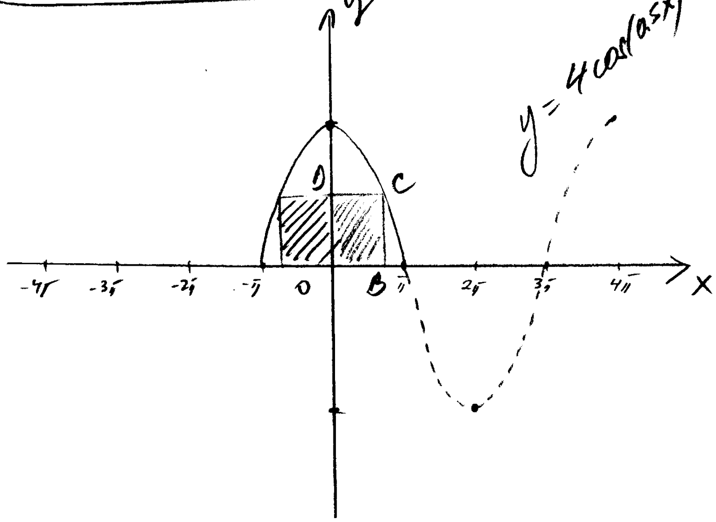
$S = x^2 + \frac{4x \cdot 500}{x^2}$

$S = x^2 + \frac{2000}{x}$, $x > 0$

Find critical points:
$S' = 2x + 2000 \cdot \frac{-1}{x^2}$
$S' = 0 \Rightarrow 2x - \frac{2000}{x^2} = 0$
$\frac{2x^3 - 2000}{x^2} = 0$
$2x^3 = 2000$
$x^3 = 1000$
$x = 10$ ft

Find if $x = 10$ is a min. or max.
$S'' = 2 - 2000(-2)x^{-3}$
$S'' = 2 + \frac{4000}{x^3}$
$S''(10) > 0$, so $x = 10$ minimizes the surface area, so the optimum dimensions are 10x10x5

PROBLEM 5



$y = 4\cos(0.5x)$ $\quad A = 4$ $\quad T = \frac{2\pi}{0.5} = 4\pi$

Area rectangle max iff Area OBCD max

let $B(x,0)$
then $C(x, 4\cos 0.5x)$

Let $A(x)$ = area of rectangle OBCD

$A(x) = x \cdot 4\cos(0.5x), \quad 0 \le x \le \pi$

$A(x) = 4x\cos(0.5x)$

Find critical points:

$A'(x) = 4\cos(0.5x) + 4x(-\sin 0.5x)0.5$

$A'(x) = 4\cos(0.5x) - 2x\sin(0.5x)$

$A'(x) = 0$

$4\cos 0.5x - 2x\sin 0.5x = 0$

$2\cos 0.5x - x\sin 0.5x = 0$

Graph $y = 2\cos 0.5x - x\sin 0.5x$ over $[0,\pi]$ and find the x-∩

$x \approx 1.72$

$A(0) = A(\pi) = 0$

$A(1.72) = 4(1.72)\cos(0.5(1.72)) \approx 4.489$

So the maximum area occurs at $x \approx 1.72$ units

and the dimensions of rectangle are $1.7 * 2 = 3.4$ by $4\cos(0.5(1.7)) = 2.64$

and the max. area is 8.98

PROBLEM 6

$s(t) = -16t^2 + 96t + 112$

s = height (ft)
t = time (sec)

(a) v = velocity $= s'(t)$

$v(t) = -32t + 96$

$v(0) = 96$ ft/sec initial velocity

(b) max. height occurs when $s'(t) = 0$

$-32t + 96 = 0$

$t = 3$ seconds

$s(3) = -16(9) + 96(3) + 112$

$s(3) = 256$ ft max. height

(c) $s(t) = 0$

$-16(t^2 - 6t - 7) = 0$

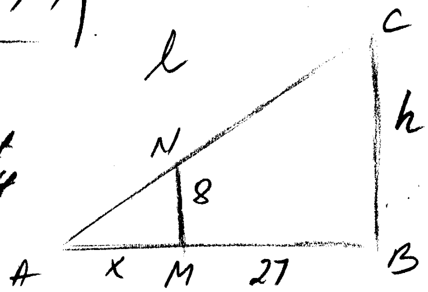
$-16(t-7)(t+1) = 0$ $\quad t = 7$, $\cancel{t = -1}$

so $s(t) = 0$ at $t = 7$ sec.

$v(7) = -32(7) + 96$

$v(7) = -128$ ft/sec

$\boxed{\text{PROBLEM 7}}$



Given
$MN = 8$ ft
$MB = 27$ ft

Let l = length of beam AC
$x = AM$
$BC = h$

Pythagorean theorem in $\triangle ABC$:

$$l^2 = (x+27)^2 + h^2$$

$\triangle AMN \sim \triangle ABC$ ($MN \parallel BC$)

$$\frac{8}{h} = \frac{x}{x+27} \Rightarrow h = \frac{8(x+27)}{x}$$

$$h = 8 + \frac{216}{x}$$

$$l^2 = (x+27)^2 + \left(8+\frac{216}{x}\right)^2, \quad x>0$$

l_{min} iff l^2 min.

Find critical points

Let $l^2 = f(x)$

$$f(x) = (x+27)^2 + \left(8+\frac{216}{x}\right)^2$$

$$f'(x) = 2(x+27) + 2\left(8+\frac{216}{x}\right)\left(\frac{-216}{x^2}\right)$$

$$f'(x) = 2(x+27) - \frac{432}{x^2}\left(8+\frac{216}{x}\right)$$

$f'(x) = 0 \Rightarrow$

$$2(x+27) - \frac{432}{x^2}\cdot\frac{8(x+27)}{x} = 0$$

$$\frac{2x^3(x+27) - 3456(x+27)}{x^3} = 0$$

$$2(x+27)(x^3-1728) = 0$$

$\cancel{x = -27}$ (not positive)

or $x^3 = 1728$, $x = 12$

check whether $x=12$ is a min. or max.

$$f''(x) = 2 - 432(-2)x^{-3}\left(8+\frac{216}{x}\right) - \frac{432}{x^2}\left(\cdot 216(-1)x^{-2}\right)$$

$$f''(x) = 2 + \frac{864}{x^3}\left(8+\frac{216}{x}\right) + \frac{93{,}312}{x^4}$$

$f''(x) > 0$, $\forall x > 0$

so $x=12$ minimizes the length of the beam

$$l_{min} = \sqrt{(12+27)^2 + \left(8+\frac{216}{12}\right)^2}$$

$$l_{min} \approx 46.87 \text{ ft.}$$

PROBLEM 8)

$s_1 = \sin t$
$s_2 = \sin(t+\frac{\pi}{3})$ position functions of the particles

ⓐ They meet iff $s_1 = s_2$

$\sin t = \sin(t+\frac{\pi}{3})$

$\sin t = \sin t \cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos t$

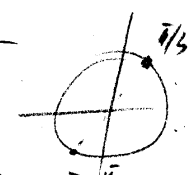
$\sin t = \sin t \cdot \frac{1}{2} + \frac{\sqrt{3}}{2}\cos t \;\Big/ \div \cos t$

note that $\cos t \neq 0$
(if it is, then $\sin t = 0$ which is impossible, since $\sin t$ and $\cos t$ can't be zero simultaneously)

$\tan t = \frac{1}{2}\tan t + \frac{\sqrt{3}}{2}$

$\frac{1}{2}\tan t = \frac{\sqrt{3}}{2}$

$\tan t = \sqrt{3}$ $\Big\langle$ $t = \frac{\pi}{3}$ or $t = \frac{4\pi}{3}$



ⓑ let $d(t)$ = distance between the particles

$d(t) = |s_1 - s_2| = |\sin t - \sin(t+\frac{\pi}{3})|$

$d(t) = |\sin t - \frac{1}{2}\sin t - \frac{\sqrt{3}}{2}\cos t|$

$d(t) = |\frac{1}{2}\sin t - \frac{\sqrt{3}}{2}\cos t|$

$d(t) = \frac{1}{2}|\sin t - \sqrt{3}\cos t|$

Find critical points

(Recall that if $f(x) = |x|$

$f(x) = \begin{cases} -x, & x<0 \\ x, & x \geq 0 \end{cases}$ $\quad f'(x) = \begin{cases} -1, & x<0 \\ 1, & x>0 \end{cases}$

$f'(x) = \frac{x}{|x|}, \; \forall x \neq 0$)

if $f(x) = |u|$
where u = fct. of x

$f(x) = \begin{cases} -u, & u<0 \\ u, & u \geq 0 \end{cases}$

$f'(x) = \begin{cases} -u \cdot u', & u<0 \\ u \cdot u', & u>0 \end{cases}$

$f'(x) = \frac{u}{|u|} \cdot u', \; \forall u \neq 0$

In our case, $u = \frac{1}{2}(\sin t - \sqrt{3}\cos t)$

$d'(x) = \frac{\frac{1}{2}(\sin t - \sqrt{3}\cos t)}{|\frac{1}{2}(\sin t - \sqrt{3}\cos t)|}\left(\frac{1}{2}(\cos t + \sqrt{3}\sin t)\right)$

$d'(x) = \frac{(\sin t - \sqrt{3}\cos t)(\cos t + \sqrt{3}\sin t)}{2|\sin t - \sqrt{3}\cos t|}$

$d'(x) = 0$ iff

I $\sin t = \sqrt{3}\cos t \;\Big/ \div \cos t \neq 0$

$\tan t = \sqrt{3}$

$t = \frac{\pi}{3}$ or $t = \frac{4\pi}{3}$

or

II $\cos t = -\sqrt{3}\sin t \;\Big/ \div \cos t \neq 0$

$\tan t = \frac{-1}{\sqrt{3}}$

$t = \frac{5\pi}{6}$ or $t = \frac{11\pi}{6}$

notice that $d'(x)$ not defined when $\sin t = \sqrt{3}\cos t$ (case I)

Evaluate $d(t)$ at

$t = 0, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{6}, \frac{11\pi}{6}, 2\pi$

$d(0) = d(2\pi) = \frac{\sqrt{3}}{2}$

$d(\frac{\pi}{3}) = 0$

$d(\frac{5\pi}{6}) = 1$

$d(\frac{4\pi}{3}) = 0$

$d(\frac{11\pi}{6}) = 1$

so the greatest distance between the particles is 1 m.

(c) The distance changes the fastest when $d'(t)$ is greatest. That will occur near where $d'(t)$ is undefined, as $d'(t)$ will have cusps there: at $t = \frac{\pi}{3}$ and $t = \frac{4\pi}{3}$

PROBLEM 9

$n = \frac{a}{x-c} + b(100-x)$

x = cost (\$)

c = unit cost (\$)

$a, b > 0$

n = number sold

let p = profit

p = revenue − cost

revenue = nx

cost = nc

$p = nx - nc = n(x-c)$

$p = \left(\frac{a}{x-c} + b(100-x)\right)(x-c)$

$p = a + b(100-x)(x-c)$

Find critical points

$p' = b(-1)(x-c) + b(100-x)(1)$

$p' = -b(x-c) + b(100-x)$

$p' = b(100 - x - x + c)$

$p' = 0$ if $100 + c - 2x = 0$

$x = \frac{100+c}{2} = 50 + \frac{c}{2}$

Check if $x = 50 + \frac{c}{2}$ is a min. or max.

$p'' = b(-2)$

$p'' = -2b < 0, \forall x$

So $x = 50 + \frac{c}{2}$ \$ in order to maximize profit

PROBLEM 10

$C(x) = x^3 - 20x^2 + 20000x$

cost of producing x

average cost $\bar{C}(x) = \frac{C(x)}{x}$

$\bar{C}(x) = x^2 - 20x + 20{,}000$

Find critical pts:

$\bar{C}'(x) = 2x - 20 = 0 \Rightarrow x = 10$

$\bar{C}''(x) = 2 > 0, \forall x$

so $x = 10$ will minimize the average cost $\bar{C}(10) = \$19{,}900$ per item when 10 items are produced