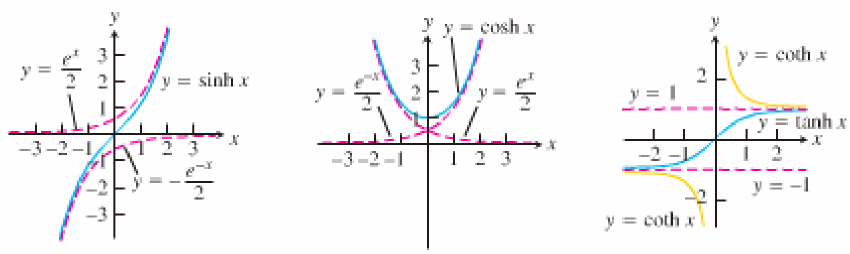
3.11 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical applications. In this section we give a brief introduction to these functions, their graphs, and their derivatives.

Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished, for the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire.

The six basic hyperbolic functions



(a) (b) (c)

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:

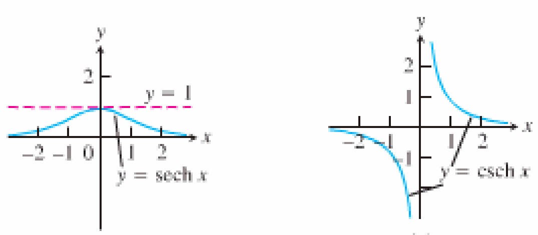
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d) (e)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Note: We pronounce sinh *x* as "cinch *x*,", rhyming with "pinch *x*,", and cosh *x* as "kosh *x*,", rhyming with "gosh *x*."

Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

then $$1 - \tanh^2 x = \text{sech}^2 x$$
$$\coth^2 x - 1 = \text{csch}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

then $$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\sinh(-x) = \sinh x$$
$$\cosh(-x) = \cosh x$$

Note: The identity $\cosh^2 u - \sinh^2 = 1$, where u is a real number, tells us that the point having coordinates $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$. This is where the *hyperbolic* functions get their names.

Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x$$

$$\frac{d}{dx}(\text{csch}\, x) = -\text{csch}\, x \coth x$$

Inverse hyperbolic functions

$\sinh : \mathbb{R} \to \mathbb{R}$ is one-to-one, then
$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$
$y = \sinh^{-1} x \Leftrightarrow \sinh y = x$

$\cosh : [0, \infty) \to [1, \infty)$ is one-to-one, then
$\cosh^{-1} : [1, \infty) \to [0, \infty)$
$y = \cosh^{-1} x \Leftrightarrow \cosh y = x$

$\tanh : \mathbb{R} \to (-1,1)$ is on-to-one, then
$\tanh^{-1} : (-1,1) \to \mathbb{R}$
$y = \tanh^{-1} x \Leftrightarrow \tanh y = x$

$\coth : (-\infty, 0) \cup (0, \infty) \to (-\infty, -1) \cup (1, \infty)$ is one-to-one, then
$\coth^{-1} : (-\infty, -1) \cup (1, \infty) \to (-\infty, 0) \cup (0, \infty)$
$y = \coth^{-1} x \Leftrightarrow \coth y = x$

$\operatorname{sech} : [0, \infty) \to (0,1]$ is one-to-one, then
$\operatorname{sech}^{-1} : (0,1] \to [0, \infty)$
$y = \operatorname{sech}^{-1} x \Leftrightarrow \operatorname{sech} y = x$

$\operatorname{csch} : (-\infty, 0) \cup (0, \infty) \to (-\infty, 0) \cup (0, \infty)$ is one-to-one, then
$\operatorname{csch}^{-1} : (-\infty, 0) \cup (0, \infty) \to (-\infty, 0) \cup (0, \infty)$
$y = \operatorname{csch}^{-1} x \Leftrightarrow \operatorname{csch} y = x$

Derivatives of inverse hyperbolic functions

$$\frac{d\left(\sinh^{-1} x\right)}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d\left(\cosh^{-1} x\right)}{dx} = \frac{1}{\sqrt{x^2-1}}, \; x > 1$$

$$\frac{d\left(\tanh^{-1} x\right)}{dx} = \frac{1}{1-x^2}, |x| < 1$$

$$\frac{d\left(\coth^{-1} x\right)}{dx} = \frac{1}{1-x^2}, |x| > 1$$

$$\frac{d\left(\operatorname{sech}^{-1} x\right)}{dx} = \frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$$

$$\frac{d\left(csh\mathrm{h}^{-1} x\right)}{dx} = \frac{1}{|x|\sqrt{1+x^2}}, x \neq 0$$

Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), -1 < x < 1$$