Section 2.6
Limits at Infinity; Horizontal Asymptotes

Definition 1 If f is a function defined on some interval, (a,∞) we write

$$\lim_{x\to\infty} f(x) = L$$

and say "the limit of $f(x)$, as x approaches ∞, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x sufficiently large.

Definition 2 If f is a function defined on some interval, $(-\infty,a)$ we write

$$\lim_{x\to-\infty} f(x) = L$$

and say "the limit of $f(x)$, as x approaches $-\infty$, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x sufficiently large negative.

Definition 3 The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if and only if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Geometric illustrations of the definition

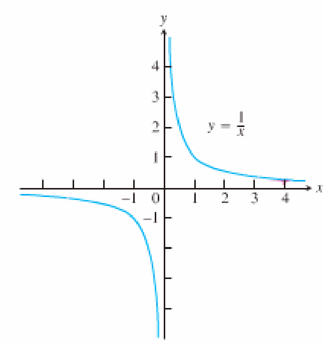

FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

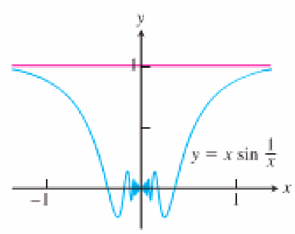

FIGURE 2.55 The line $y = 1$ is a horizontal asymptote of the function

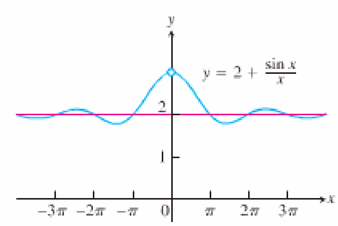

FIGURE 2.57 A curve may cross one of its asymptotes infinitely often (Example 8).

Note: All the Limit Laws (sum, difference, constant multiple, product, quotient, power, and root) are true when we replace $\lim_{x\to a}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$.

Exercise 1 Find the limit or show that it does not exist:

a) $\lim\limits_{x\to\infty}\dfrac{10x^5+x^4+31}{x^6}$

b) $\lim\limits_{x\to-\infty} e^x$

c) $\lim\limits_{x\to\infty}\sin\dfrac{1}{x}$

d) $\lim\limits_{t\to\infty}\dfrac{2+\sqrt{t}}{2-\sqrt{t}}$

e) $\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{x}-5x+3}{2x+x^{\frac{2}{3}}-4}$

f) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

g) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

h) $\lim\limits_{q\to-\infty}\dfrac{\cos q}{3q}$

i) $\lim\limits_{t\to-\infty}\dfrac{2-t+\sin t}{t+\cos t}$

j) $\lim\limits_{x\to\infty} e^{-x}\sin x$

k) $\lim\limits_{x\to\pm\infty} x\sin\dfrac{1}{x}$

l) $\lim\limits_{t\to-\infty}\dfrac{\cos\frac{1}{t}}{1+\frac{1}{t}}$

m) $\lim\limits_{x\to-\infty}\dfrac{4x^3+6x^2-2}{2x^3-4x+5}$

n) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{1+4x^6}}{2-x^3}$

o) $\lim\limits_{x\to\infty}\dfrac{x+3x^2}{4x-1}$

p) $\lim\limits_{x\to\infty}\left(e^{-x}+2\cos 3x\right)$

r) $\lim\limits_{x\to\infty}\dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$

s) $\lim\limits_{q\to\infty}\dfrac{\sin^2 q}{q^2+1}$

t) $\lim\limits_{x\to\infty}\left(x+x\sin\dfrac{1}{x}\right)$

Answers :a)0; b)0; c)0; d) -1; e)-5/2; f) 1; g) -1; h) 0; i) -1; j) 0; k) 1; l) 1; m) 2; n) 2; o) ∞ ; p) DNE; r) 1; s) 0; t) ∞

Exercise 2 Find the following limits:

a) $\lim\limits_{x\to1^+}\left(\dfrac{1}{x-1}-\dfrac{1}{x^3-1}\right)$

b) $\lim\limits_{x\to1^+}\left(\dfrac{1}{x-1}-\dfrac{3}{x^3-1}\right)$

c) $\lim\limits_{t\to\infty}\left(t-\sqrt{t^2-1}\right)$

d) $\lim\limits_{t\to\infty}\left(t-\sqrt{t^2+5t}\right)$

e) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+3}+x\right)$

f) $\lim\limits_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)$

g) $\lim\limits_{x\to\infty}\left[\ln(2+x)-\ln(1+x)\right]$

Answers: a) ∞ ; b) 1; c) 0; d) -5/2; e) 0; f) -3/4 ; g) 0

Exercise 3 Find the following limits:

a) $\lim\limits_{x\to\infty}\sqrt{x^2+1}$

c) $\lim\limits_{x\to\infty}\dfrac{x^2}{\sqrt{x^4+1}}$

d) $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2+1}}$

e) $\lim\limits_{x\to-\infty}\dfrac{x}{\sqrt{x^2+1}}$

Answers; a) ∞ ; c) 1; d) 1; e) -1

Exercise 4 Use limits to determine the equations for all vertical and horizontal asymptotes:

a) $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x + 1}$ b) $g(x) = \dfrac{\sqrt{x} + 4}{\sqrt{x+4}}$

Exercise 5 a) A tank contains 5000L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Find a formula that gives the concentration of salt after *t* minutes (in grams per liter).
b) What happens to the concentration as $t \to \infty$?

Exercise 6 Find the limits as $x \to \infty$ and $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph.

$$f(x) = 2x^3 - x^4$$

<u>Solutions</u>