Section 2.5 - Continuity

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

Notes about intervals:

- $[a,b]$ - closed interval
- $(a,b),(a,\infty),(-\infty,b)$ - open intervals
- If f is defined on $[a,b]$, we say that c is an **interior point** if and only if $c \in (a,b)$.

We define continuity at a point in a function's domain.

Definition 1 1) A function f is **continuous at an interior point *c*** of its domain if and only if

$$\lim_{x\to c} f(x) = f(c).$$

2) A function f is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if and only if

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b)\text{, respectively.}$$

Note: The definition requires three things to happen in order for a function f to be continuous at a point c:

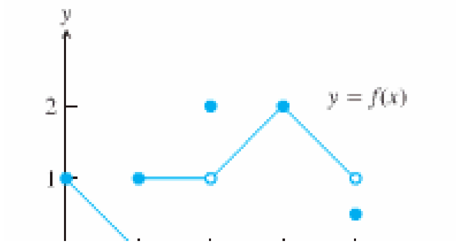
1. $f(c)$ is defined (that is, c is in the domain)
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

Definition 2 If f is not continuous at c, we say that f is **discontinuous at *c***, or f has a **discontinuity** at c.

Notes:

- The definition says that f is continuous at c if $f(x)$ approaches $f(c)$ as x approaches c. Thus, a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we like by keeping the change in x sufficiently small.
- Geometrically, we can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing the pen from the paper.

Example 1 Find the points at which the function f in the figure is continuous and the points at which f is not continuous.



How to detect discontinuities when a function is defined by a formula?

Example 2 a) $f(x)=\dfrac{x^2-x-2}{x-2}$ b) $f(x)=\begin{cases}\dfrac{1}{x^2}, & x\neq 0\\ 1, & x=0\end{cases}$

Definition 3 A function f is **continuous from the right** at a point c if and only if $\lim_{x\to c^+} f(x)=f(c)$.

A function f is **continuous from the left** at a point c if and only if $\lim_{x\to c^-} f(x)=f(c)$.

A function is **continuous on an interval** if and only if it is continuous at every number in the interval. (we understand continuity at an endpoint to mean continuity from the right or left)

Example 3 Show that the function $f(x)=1-\sqrt{1-x^2}$ is continuous on the interval $[-1,1]$.

Theorem – Properties of Continuous Functions

If the functions f and g are continuous at c, then the following functions are continuous at c:

$f+g, f-g$

$k\cdot f$, for any $k\in\mathbb{R}$

fg

$\dfrac{f}{g}, g(c)\neq 0$

$f^n, n\in\mathbb{N}$

$\sqrt[n]{f}, n\in\mathbb{N}$, provided that f is defined on an open interval containing c

Theorem

Any polynomial function is continuous on $(-\infty,\infty)$.

Any rational function is continuous on its domain.

Theorem

> The following types of functions are continuous at every point in their domain:
>
> | | | |
> |---|---|---|
> | polynomials | rational functions | root functions |
> | trigonometric functions | inverse trigonometric functions | |
> | exponential functions | logarithmic functions | |

Example 4 Where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Theorem – Composition of Continuous Functions

> If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Theorem – Limits of Continuous Functions

> If g is continuous at a point b and $\lim\limits_{x \to c} f(x) = b$, then
>
> $$\lim_{x \to c} (g \circ f)(x) = \lim_{x \to c} g\left(f(x)\right) = g\left(\lim_{x \to c} f(x)\right)$$

Example 5 Evaluate $\lim\limits_{x \to 1} \sin^{-1}\left(\dfrac{1-\sqrt{x}}{1-x}\right)$.

Example 6 Evaluate $\lim\limits_{x \to 0} \sqrt{x+1} \cdot e^{\tan x}$.

Continuous Extension to a Point

Example 7 Let $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ ?\quad, & x = 0 \end{cases}$.

Can we extend the function's domain to include $x = 0$ such that the extended function is continuous?

Exercise 1 Find the following limits:

a) $\lim\limits_{x \to p} \sin\left(x - \sin x\right)$

b) $\lim\limits_{x \to 0} \tan\left(\frac{p}{4} \cdot \cos\left(\sin x^{\frac{1}{3}}\right)\right)$

Exercise 2 For what values of a and b is the given function continuous?

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x < 2 \\ 2ax^2 - bx + 3, & 2 \le x < 3 \\ 2x - a + b, & x \ge 3 \end{cases}$$

Exercise 3 Let $f(x) = \begin{cases} x + 2, & x < 0 \\ e^x, & 0 \le x \le 1 \\ 2 - x, & x > 1 \end{cases}$

a) Find the numbers at which the function is discontinuous.
b) At which of these points is f continuous from the right, from the left, or neither?
c) Sketch the graph of f.

Exercise 4 The gravitational force exerted by Earth on a unit mass at a distance r from the center of the planet is

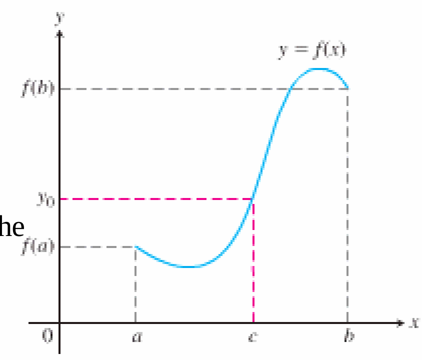
$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \ge R \end{cases}.$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

Theorem A function f is continuous at a if and only if $\lim_{h \to 0} f(a+h) = f(a)$.

Theorem – The Intermediate Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a,b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a,b]$.



Notes:

- The Intermediate Value Theorem states that a continuous function takes on every intermediate value between $f(a)$ and $f(b)$.
- Geometrically, the Theorem says that any horizontal line crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the graph at least once in the interval $[a,b]$
- One use of the Theorem is in locating roots of equations.

Exercise 5 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Exercise 6 Prove that sine is a continuous function.

Exercise 7 a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.

b) Prove that if f is a continuous function on an interval, then so is $|f|$.

c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

<u>Solutions</u>