M 180

SOLUTIONS

REVIEW - SELECTED EXERCISES

(1b) $f(r) = \dfrac{r}{r^2+1}$, $r \in \mathbb{R}$

$f'(r) = \dfrac{-r^2+1}{(r^2+1)^2}$

$f'(r) = 0$ iff $1 - r^2 = 0$, $r = \pm 1$

$f'(r)$ defined for $\forall r$

critical numbers : $\boxed{r = \pm 1}$

(1c) $f(z) = \dfrac{z+1}{z^2+z+1}$, $z \in \mathbb{R}$ $(z^2+z+1 \neq 0)$

$f'(z) = \dfrac{-z^2-2z}{(z^2+z+1)^2}$

$f'(z) = 0$ iff $-z^2 - 2z = 0$

$-z(z+2) = 0$ $\left\langle \begin{array}{l} z=0 \\ z=-2 \end{array}\right.$

$f'(z)$ is defined for $\forall z$

$(z^2+z+1 \neq 0)$

critical numbers : $\boxed{z = 0, z = -2}$

(1d) $f(x) = x^{\frac{4}{5}}(x-4)^2$, $x \in \mathbb{R}$

$f'(x) = \frac{4}{5} x^{-\frac{1}{5}}(x-4)^2 + x^{\frac{4}{5}} \cdot 2(x-4)(1)$

$= x^{-\frac{1}{5}}(x-4)\left(\frac{4}{5}(x-4) + 2x\right)$

$= \dfrac{x-4}{x^{\frac{1}{5}}} \; \dfrac{4(x-4)+10x}{5} = \dfrac{(x-4)(14x-16)}{5x^{\frac{1}{5}}}$

$f'(x) = \dfrac{2(x-4)(7x-8)}{5x^{\frac{1}{5}}}$

$f'(x) = 0$ iff $(x-4)(7x-8) = 0$

$x = 4$, $x = \frac{8}{7}$

$f'(x)$ undefined iff $x = 0$

Critical numbers : $\boxed{x = 4, x = \frac{8}{7}, x = 0}$

(1e) $F(x) = \sqrt[3]{x^2 - x} = (x^2-x)^{\frac{1}{3}}$ $x \in \mathbb{R}$

$F'(x) = \dfrac{2x-1}{3(x^2-x)^{2/3}}$

$F'(x) = 0$ iff $2x - 1 = 0$, $x = \frac{1}{2}$

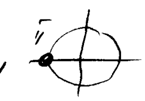
$F'(x)$ undefined iff $x^2 - x = 0$

$x = 0, x = 1$

Critical numbers : $\boxed{x = \frac{1}{2}, x = 0, x = 1}$

(1g) $g(\theta) = \theta + \sin\theta$, $\theta \in \mathbb{R}$

$g'(\theta) = 1 + \cos\theta$



$g'(\theta) = 0$ iff $\cos\theta = -1$

$\theta = \pi + 2\pi k$

$\theta = \pi(1 + 2k)$, $k \in \mathbb{Z}$

Critical #'s : $\boxed{\theta = \pi(1+2k)}$ $k \in \mathbb{Z}$

(1h) $F(x) = x \ln x$, $x > 0$

$f'(x) = \ln x + 1$

$f'(x) = 0$ iff $\ln x = -1$

$x = e^{-1} = \frac{1}{e}$

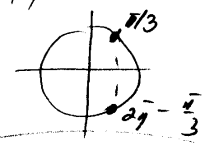
Critical number $\boxed{x = \frac{1}{e}}$

(2a) $f(x) = x - 2\sin x$, $x \in [0, 2\pi]$.

1st. Find critical numbers

$f'(x) = 1 - 2\cos x$

$f'(x) = 0$ iff $\cos x = 1/2$ $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$



CP: $x = \frac{\pi}{3}$, $x = \frac{5\pi}{3}$

2nd $f(0) = 0$

$f(2\pi) = 2\pi \approx 6.28$

$f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3} \approx -0.68$

$f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$

Absolute min. value is $f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3}$

Absolute max. value is $f(\frac{5\pi}{3}) = \frac{5\pi}{3} + \sqrt{3}$

(2c) $f(x) = x^2 + \frac{2}{x}$, $x \in [\frac{1}{2}, 2]$

1st Find the critical #s

$f'(x) = 2x - \frac{2}{x^2} = \frac{2(x^3 - 1)}{x^2}$

$f'(x) = 0$ iff $x^3 - 1 = 0$

$(x-1)(x^2 + x + 1) = 0$

$x = 1$ $(x^2 + x + 1 \neq 0)$

$f'(x)$ is undefined iff $x = 0$, but $0 \notin [\frac{1}{2}, 1]$

CP: $x = 1$

2nd $f(\frac{1}{2}) = \frac{17}{4} = 4.25$

$f(1) = 3$

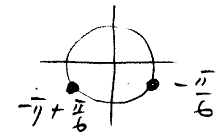
$f(2) = 5$

Abs. min value is $f(1) = 3$

Abs. max value is $f(2) = 5$

(2e) $f(x) = x - 2\cos x$, $x \in [-\pi, \pi]$

1st critical numbers:



$f'(x) = 1 + 2\sin x$

$f'(x) = 0$ iff $\sin x = -\frac{1}{2}$

CP: $x = -\frac{\pi}{6}$, $x = \frac{-5\pi}{6}$

2nd $f(-\pi) = 2 - \pi \approx -1.14$

$f(\pi) = \pi + 2 \approx 5.14$

$f(-\frac{\pi}{6}) = -\frac{\pi}{6} - \sqrt{3} \approx -2.26$

$f(\frac{-5\pi}{6}) = \sqrt{3} - \frac{5\pi}{6} \approx -0.886$

Abs. max is $f(\pi) = \pi + 2$

Abs. min. is $f(-\frac{\pi}{6}) = -\frac{\pi}{6} - \sqrt{3}$

(3b) $f(x) = \dfrac{x}{(1+x)^2}$

x	$-\infty$		-1		0		1		2		∞
f'	$-$	$-$	$\vert$	$+$	$+$	$+$	0	$-$	$-$	$-$	
f	H.A. $y=0$	$\searrow$	$-\infty \vert -\infty$	$\nearrow$	0	$\nearrow$	$\frac{1}{4}$	$\searrow$	$\frac{2}{9}$	$\searrow$	H.A. $y=0$
f''	$-$	$-$	$\vert$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$

(f) Domain: $x \in \mathbb{R} \setminus \{-1\}$

$\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{x}{(1+x)^2} = \lim\limits_{\pm\infty} \dfrac{\frac{x}{x^2}}{(\frac{1}{x}+1)^2} = \dfrac{0}{0+1} = 0$

H.A. $y=0$

$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{1^-} \dfrac{x}{(1+x)^2} = \dfrac{-1}{0^+} = -\infty$

$\lim\limits_{x\to -1^+} f(x) = \dfrac{-1}{0^+} = -\infty$ $\Big\} \to$ V.A. $x = 1$

x-in: $y=0$ iff $x=0$

(f') $f'(x) = \dfrac{1-x}{(1+x)^3}$

$f'(x)=0$ iff $x=1$

sign of $f'(x)$: TP: $x=-10$, $y = \frac{+}{-} = -$

TP: $x=0$, $y=+$

TP: $x=10$, $y = \frac{-}{+} = -$

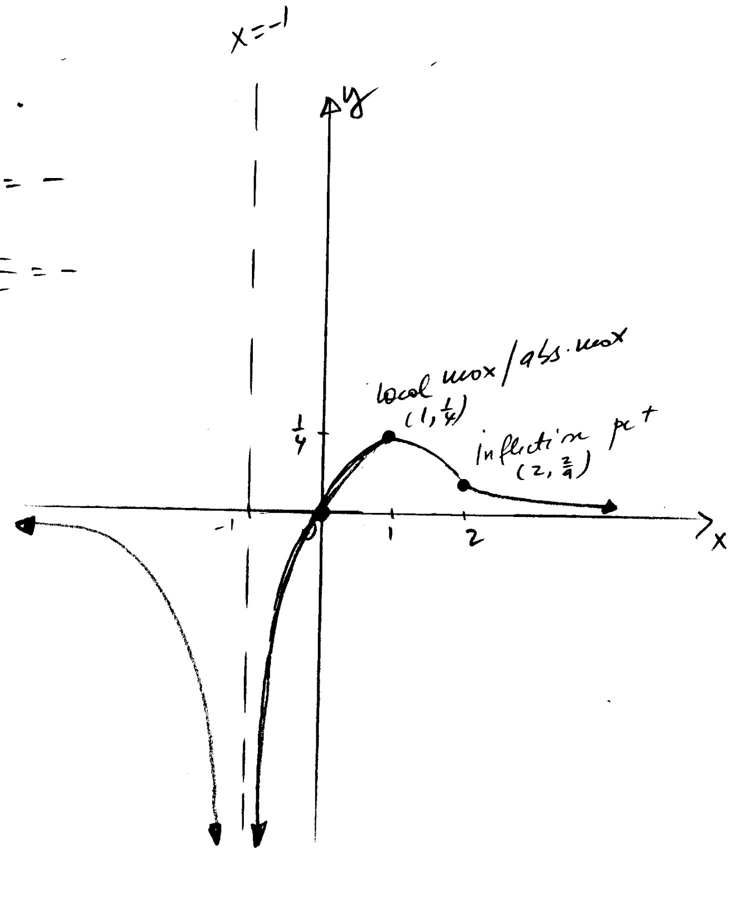
$f(1) = \dfrac{1}{2^2} = \dfrac{1}{4}$

(f'') $f''(x) = \dfrac{2x-4}{(1+x)^4}$

$f''(x)=0$ iff $2x-4=0$, $x=2$

The sign of f'' is given by the sign of $y=2x-4$

$f(2) = \dfrac{2}{3^2} = \dfrac{2}{9}$



(3c) $f(x) = \dfrac{\ln x}{\sqrt{x}}$

x	0		1		e^2		$e^{\frac{8}{3}}$		∞
f'		$+ \; +$		$+ \; +$	0	$- \; - \; -$		$- \; - \; - \; -$	
f	$-\infty$	$\nearrow$	0	$\nearrow$	$\frac{2}{e}$	$\searrow$	$\frac{8}{3e^{4/3}}$	$\searrow$	H.A. $y=0$
f''		$- \; -$		$- \; - \; -$		$- \; -$	0	$+ \; + \; + \; +$	

(f) Domain : $x > 0$

$$\lim_{x\to\infty} f(x) = \lim_{\infty} \frac{\ln x}{\sqrt{x}} = \frac{\infty}{\infty} \text{ (l'Hopital)}$$

$$= \lim_{\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{\infty} \frac{2\sqrt{x}}{x} = \lim_{\infty} \frac{2}{\sqrt{x}} = \frac{2}{\infty} = 0$$

H.A. $y = 0$

$$\lim_{x\to 0^+} f(x) = \lim_{0^+} \frac{\ln x}{\sqrt{x}} = \lim_{0^+} \frac{1}{\sqrt{x}} \cdot \ln x = \frac{1}{0^+} \cdot (-\infty) = +\infty(-\infty) = -\infty$$

V.A. $x = 0$

x-n: $y = 0$ iff $\ln x = 0 \Rightarrow x = 1$

(f') $f'(x) = \dfrac{\frac{1}{\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}}{x} = \dfrac{2 - \ln x}{2x\sqrt{x}}$

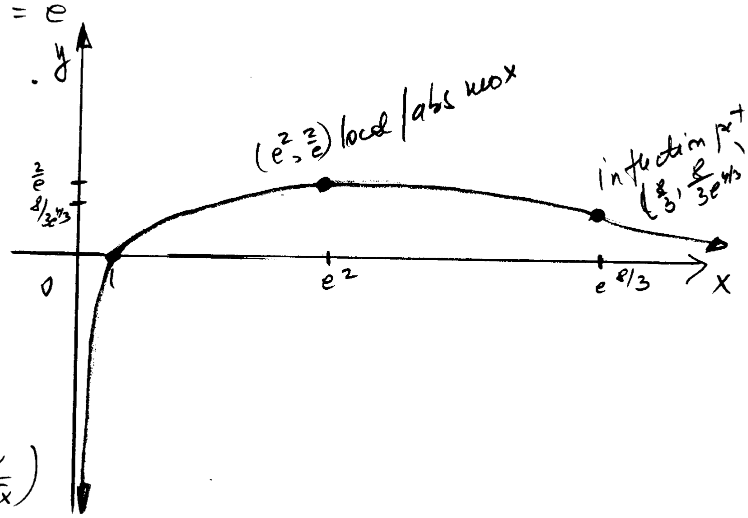
$f'(x) = 0$ iff $\ln x = 2 \Rightarrow x = e^2$

The sign of f' is given by the sign of $y = 2 - \ln x$, as $2x\sqrt{x} > 0 \quad \forall x > 0$

TP: $x = 1$, $y = 2 - \ln 1 > 0$

TP: $x = e^3$, $y = 2 - \ln e^3 = 2 - 3 < 0$

$f(e^2) = \dfrac{\ln e^2}{\sqrt{e^2}} = \dfrac{2}{e}$



(f'') $f''(x) = \dfrac{-\frac{1}{x} \cdot 2x\sqrt{x} - (2 - \ln x)\left(2\sqrt{x} + \frac{2x}{2\sqrt{x}}\right)}{4x^2 \cdot x}$

$$f''(x) = \frac{-2\sqrt{x} - (2 - \ln x) \cdot 3\sqrt{x}}{4x^3} = \frac{\sqrt{x}\,(3\ln x - 8)}{4x^3}$$

$f''(x) = 0$ iff $3\ln x - 8 = 0$

$\ln x = \frac{8}{3}$, $x = e^{\frac{8}{3}}$

$f(e^{\frac{8}{3}}) = \dfrac{8}{3e^{4/3}}$

The sign of f'' is given by the sign of $y = 3\ln x - 8$ (as $\frac{\sqrt{x}}{4x^3} > 0$, $\forall x > 0$)

(4b) $\lim_{x\to 1} \frac{x^a-1}{x^b-1} = \frac{0}{0}$ (l'Hopital)

$= \lim_{x\to 1} \frac{(x^a-1)'}{(x^b-1)'} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}}$

$= \frac{a}{b}$

(4d) $\lim_{x\to 0} \frac{\tan px}{\tan qx} = \frac{0}{0}$ (l'Hopital)

$= \lim_{0} \frac{p\sec^2 px}{q\sec^2 qx} = \frac{p\cdot \sec^2(0)}{q\cdot \sec^2(0)}$

$= \frac{p\cdot 1^2}{q\cdot 1^2} = \frac{p}{q}$

(4h) $\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2} = \frac{1-1}{0} = \frac{0}{0}$ (l'H)

$= \lim_{x\to 0} \frac{-m\sin mx + n\sin nx}{2x} = \frac{0}{0}$ (l'H)

$= \lim_{0} \frac{-m^2\cos mx + n^2\cos nx}{2} =$

$= \frac{1}{2}(-m^2\cos 0 + n^2\cos 0)$

$= \frac{1}{2}(n^2 - m^2)$

(4i) $\lim_{\infty} e^{-x}\ln x = e^{-\infty}\infty = 0\cdot\infty$

$= \lim_{\infty} \frac{\ln x}{e^x} = \frac{\infty}{\infty}$ (l'Hopital)

$= \lim_{\infty} \frac{\frac{1}{x}}{e^x} = \lim_{\infty} \frac{1}{xe^x}$

$= \frac{1}{\infty} = 0$

(j) $\lim_{\infty} x^{\frac{\ln 2}{1+\ln x}}$

let $f(x) = x^{\frac{\ln 2}{1+\ln x}} = e^{\ln f(x)}$

$\ln f(x) = \ln\left(x^{\frac{\ln 2}{1+\ln x}}\right)$

$= \frac{\ln 2}{1+\ln x}\ln x$

$\lim_{\infty} \ln y = \lim_{\infty} \frac{\ln 2\ln x}{1+\ln x} = \frac{\infty}{\infty}$ (l'H)

$= \ln 2 \lim_{\infty} \frac{\frac{1}{x}}{\frac{1}{x}} = \ln 2$

so, $\lim_{\infty} x^{\frac{\ln 2}{1+\ln x}} = \lim_{\infty} e^{\ln f(x)}$

$= e^{\lim_{\infty}\ln f(x)} = e^{\ln 2} = 2$

(4m) $\lim_{x\to\infty}\left(xe^{\frac{1}{x}} - x\right) = \infty\cdot e^0 - \infty$
$= \infty - \infty$

$= \lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right) = \infty\cdot 0$

$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}-1}{\frac{1}{x}} = \frac{0}{0}$ (l'Hopital)

$= \lim_{\infty} \frac{\left(e^{\frac{1}{x}}-1\right)'}{\left(\frac{1}{x}\right)'} =$

$= \lim_{\infty} \frac{-\frac{1}{x^2}e^{\frac{1}{x}}}{\frac{-1}{x^2}}$

$= \lim_{\infty} e^{\frac{1}{x}} = e^0 = 1$

(40) $\lim_{x\to 1^+} (x-1)\tan\frac{\pi x}{2} = 0\cdot\infty$

$= \lim_{x\to 1^+} \frac{x-1}{\cot\frac{\pi x}{2}} = \frac{0}{0}$ (l'Hopital)

$= \lim_{x\to 1^+} \frac{1}{-\csc^2\frac{\pi x}{2}\cdot\frac{\pi}{2}} = \frac{-2}{\pi}\lim_{1^+}\frac{1}{\csc^2\frac{\pi}{2}}$

$= \frac{-2}{\pi}\cdot\frac{1}{1^2} = \frac{-2}{\pi}$

(41) $\lim_{x\to 0^+} (-\ln x)^x = (+\infty)^0$

let $f(x) = (-\ln x)^x = e^{\ln f(x)}$

$\ln f(x) = \ln(-\ln x)^x$

$= x\ln(-\ln x)$

$\lim_{0^+}\ln f(x) = \lim_{0^+} x\ln(-\ln x) = 0\cdot\infty$

$= \lim_{0^+}\frac{\ln(-\ln x)}{\frac{1}{x}} = \frac{\infty}{\infty}$ (l'H)

$= \lim_{0^+}\frac{(\ln(-\ln x))'}{(\frac{1}{x})'}$

$= \lim_{0^+}\frac{\frac{1}{-\ln x}\cdot\frac{-1}{x}}{-\frac{1}{x^2}}$

$= \lim_{0^+}\frac{-x^2}{x\ln x} = \lim_{0^+}\frac{-x}{\ln x}$

$= \lim_{x\to 0^+}\frac{1}{\ln x}(-x) = \frac{1}{-\infty}(0)$

$= 0\cdot 0 = 0$

so, $\lim_{x\to 0^+}(-\ln x)^x =$

$= \lim_{0^+} e^{\ln f(x)}$

$= e^{\lim_{0^+}\ln f(x)}$

$= e^0 = 1$