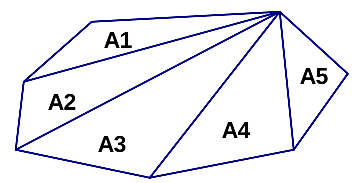
A Preview Of Calculus & 2.1 Rates of Change

Calculus is one of the greatest achievements of the human intellect. Inspired by problems in astronomy, Newton and Leibniz developed the ideas of calculus 300 years ago. Since then, each century has demonstrated the power of calculus to illuminate questions in mathematics, the physical sciences, engineering, and the social and biological sciences.

Calculus is fundamentally different from the mathematics you have studied previously. Calculus is less static and more dynamic. It is concerned with *change* and *motion*; it deals with *quantities that approach other quantities*. For that reason, it may be useful to have an overview of the subject before beginning its intensive study. Here is a glimpse of some of the main ideas of calculus by showing how the concept of a limit arises when we attempt to solve a variety of problems.

The Area Problem

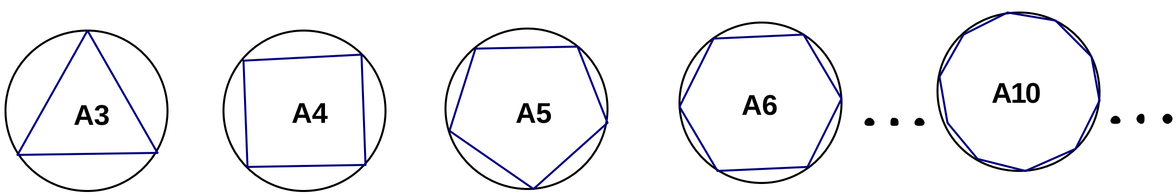
The origins of calculus go back at least 2500 years to the ancient Greeks, who found areas using the "method of exhaustion." They knew how to find the area A of any polygon by dividing it into triangles as in Figure 1 and adding the areas of these triangles.



$$A = A_1 + A_2 + A_3 + A_4 + A_5$$

It is a much more difficult problem to find the area of a curved figure. The Greek method of exhaustion was to inscribe polygons in the figure and circumscribe polygons about the figure and then let the number of sides of the polygons increase.

Figure 2 illustrates this process for the special case of a circle with inscribed regular polygons.



If A_n is the area of the inscribed polygon with n sides and A is the area of the circle, then

$$A = \lim_{n\to\infty} A_n .$$

The Greeks themselves did not use limits explicitly. However, by indirect reasoning, Eudoxus (fifth century B.C.) used exhaustion to prove the familiar formula for the area of a circle: $A = \pi r^2$.

We will use a similar idea in Chapter 5 to find areas of regions under a curve.

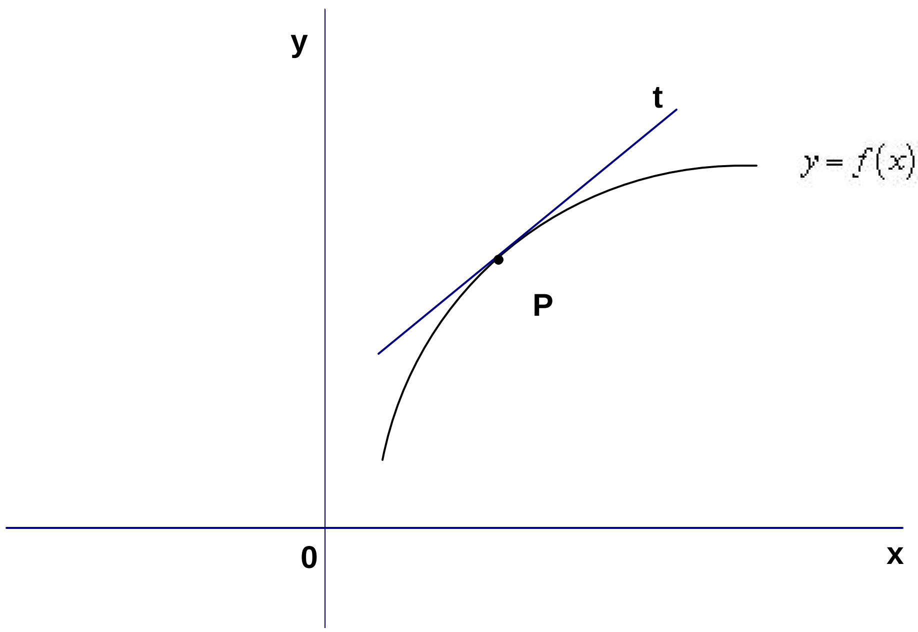
The **area problem** is the central problem in the branch of calculus called ***integral* calculus**.
The techniques that we will develop in Chapter 5 for finding areas will also enable us to compute the volume of a solid, the length of a curve, the force of water against a dam, the mass and center of gravity of a rod, and the work done in pumping water out of a tank.

The Tangent Problem

The word *tangent* comes from the Latin word *tangens*, which means "touching". Therefore, a tangent to a curve is a line that touches the curve. How can this idea be made precise?

Consider the problem of trying to find an equation of the tangent line t to a curve with equation $y = f(x)$ at a given point P. We can think of it as a line that touches the curve at P. Since we know that the point P lies on the tangent line, we can find the equation of t if we know its slope m. The problem is that we need two points to compute the slope and we know only one point, P, on t.
To get around the problem we find an approximation to m by taking a nearby point Q on the curve and computing the slope m_{PQ} of the secant line PQ. [A *secant* line, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

$$m_{PQ} =$$



Now imagine that Q moves along the curve toward P. We can see that the secant line rotates and approaches the tangent line as its limiting position. This means that the slope m_{PQ} of the secant line becomes closer and closer to the slope of the tangent line. We write

$$m = \lim_{Q \to P} m_{PQ}$$

and we say that m is the limit of m_{PQ} as Q approaches P along the curve.

The **tangent problem** has given rise to the branch of calculus called ***differential* calculus**, which was not invented until more than 2000 years after integral calculus. The main ideas behind differential calculus are due to the French mathematician Pierre Fermat (1601 – 1665) and were developed by the English mathematicians John Wallis (1616 – 1703), Isaac Barrow (1630 – 1677), and Isaac Newton (1642 – 1727) and the German mathematician Gottfried Leibniz (1646 – 1716).

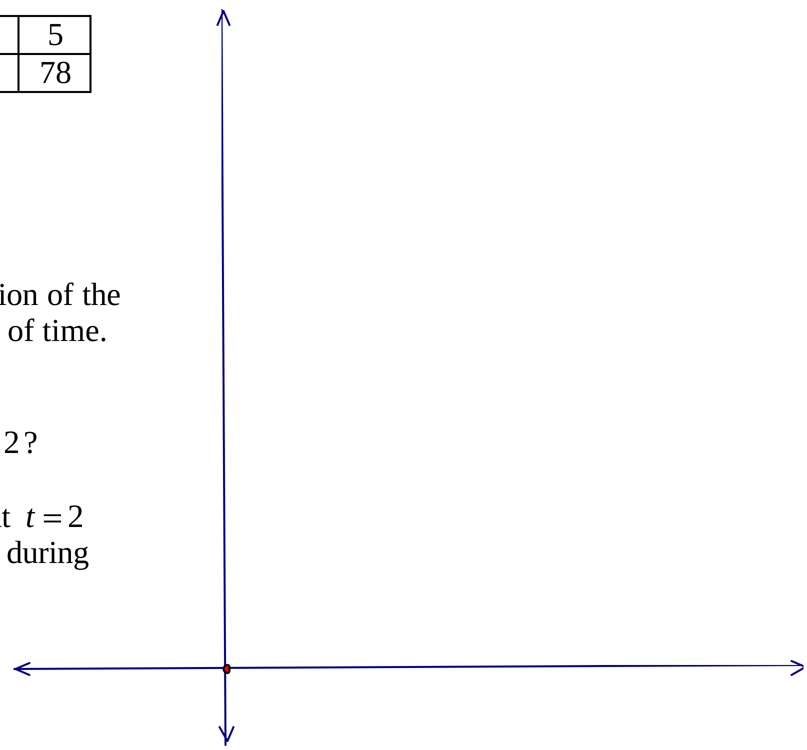
The two branches of calculus and their chief problems, the area problem and the tangent problem, appear to be very different, but it turns out that there is a very close connection between them. The tangent problem and the area problem are inverse problems in a sense that will be described in Chapter 5.

Velocity

When we look at the speedometer of a car and read that the car is traveling at 48 mi/h, what does that information indicate to us? We know that if the velocity remains constant, then after an hour we will have traveled 48 mi. But if the velocity of the car varies, what does it mean to say that the velocity at a given instant is 48 mi/h?

In order to analyze this question, let's examine the motion of a car that travels along a straight road and assume that we can measure the distance traveled by the car (in feet) at 1-second intervals as in the following chart:

t = Time elapsed (s)	0	1	2	3	4	5
d = Distance (ft)	0	2	10	25	43	78



First, show a graphical representation of the motion of the car by plotting the distance traveled as a function of time.

How do we find the instantaneous velocity at $t = 2$?

We have the feeling that the velocity at the instant $t = 2$ can't be much different from the average velocity during a short time interval starting at $t = 2$.

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

Let's calculate the average velocity in time intervals $[2, t]$ for different values of t.

Let's imagine that the distance traveled has been measured at 0.1-second time intervals as in the following chart:

t	2.0	2.1	2.2	2.3	2.4	2.5
d	10.00	11.02	12.16	13.45	14.96	16.80

$[2, 2.5]$: average velocity is

$[2, 2.4]$: average velocity is

$[2, 2.3]$: average velocity is

$[2, 2.2]$: average velocity is

$[2, 2.1]$: average velocity is

The average velocities over successively smaller intervals appear to be getting closer to ____________________ and we expect that the velocity at exactly $t = 2$ is about ____________________.

We will define instantaneous velocity of a moving object as the limiting value of the average velocities over smaller and smaller time intervals.

If we write $d = f(t)$, then $f(t)$ is the number of feet traveled after t seconds. The average velocity in the time interval $[2, t]$ is

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}} =$$

which is the same as the ____

The velocity v when $t = 2$ is the limiting value of this average velocity as t approaches 2; that is,

$v =$ ________________________________

Thus, when we solve the tangent problem in differential calculus, we are also solving problems concerning velocities. The same technique also enables us to solve problems involving rates of change in all of the natural and social sciences.

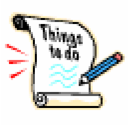
Study Examples 1 through 3 in Section 2.1 .

Sir Isaac Newton invented his version of calculus to explain the motion of the planets around the Sun. Today calculus is used in :

- Calculating the orbits of satellites and spacecraft
- Predicting population sizes
- Estimating how fast coffee prices rise
- Forecasting weather
- Measuring the cardiac output of the heart
- Calculating life insurance premiums
- A great variety of other areas

Here is a list of some of the questions that you will be able to answer using calculus:

- How can we explain the shapes of cans on supermarket shelves?
- Where is the best place to sit in a movie theater?
- How far away from an airport, should a pilot start descent?
- Where should an infielder position himself to catch a baseball thrown by an outfielder and relay it to home plate?
- Does a ball thrown upward take longer to reach its maximum height or to fall back to its original height?



Do the following exercises:

Find the average rate of change of the function over the given interval or intervals.

1. $f(x) = x^3 + 1$

   a. $[2,3]$ b. $[-1,1]$

2. $g(x) = \sin x$, $x \in \left[\frac{p}{6}, \frac{p}{3}\right]$

3. $h(x) = x^2$

   a. $[-1,1]$ b. $[-2,0]$

4. $l(t) = \cot t$, $t \in \left[\frac{p}{6}, \frac{p}{2}\right]$

Calculate the difference quotient $\frac{f(x+h) - f(x)}{h}$ for the given functions , at the given values. Show the geometric interpretation – that is, sketch a graph and explain the meaning of the difference quotient found.

5. $f(x) = x^2 - 3$ when $x = 2$.

6. $g(x) = 4x - x^2$ when $x = 1$.