3.2 Properties of Division
3.3 Zeros of Polynomials
3.4 Complex and Rational Zeros of Polynomials

In these sections we will study polynomials algebraically. Most of our work will be concerned with finding the solutions of polynomial equations of any degree – that is, equations of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0 \qquad (1)$$

Definition (3.3)

A **root** or **solution** of equation (1) is a number k that when substituted for x leads to a true statement. Thus, k is a root of equation (1) provided $f(k) = 0$.
We also refer to the number k in this case as **a zero of the function *f*.**

Note that each **real zero** is an ***x*-intercept of the graph of *f*.** We are going to find answers to the following questions:

a) How many zeros of $f(x)$ are real? Imaginary?
b) How many real zeros of $f(x)$ are positive? Negative?
c) How many real zeros of $f(x)$ are rational? Irrational?
d) Are the real zeros of $f(x)$ large or small in value?

Exercise #1 Checking for a zero or root.

a) Is -1 a zero of $P(x) = -x^3 + x^2 - x + 1$?
b) Is $x = \frac{1}{2}$ a root of the equation $2x^2 - 3x + 1 = 0$?

Note: If a root is repeated *n* times, we call it a **root of multiplicity *n***

Exercise #2 a) State the multiplicity of each root of the equation: $x^2(x+1)^3(x-1) = 0$

b) Find all zeros and their multiplicities: $f(x) = 5x^2\left(x+1-\sqrt{2}\right)(2x+5)$

c) Find all zeros and their multiplicities: $f(x) = (7x-2)^3\left(x^2+9\right)^2$

Division of Polynomials

The process of long division for polynomials follows the same four-step cycle used in ordinary long division of numbers: divide, multiply, subtract, bring down.
Notice that in setting up the division, we write both the dividend and divisor in decreasing powers of *x*.

Exercise #3 Divide $5x^3 - 6x^2 - 28x - 2$ by $x+2$.

The result of the division can be written as: ____

or

____

Note 1) Second equation is valid for all real numbers *x*, whereas first equation carries implicit restrictions that *x* my not equal -2. For this reason, we often prefer to write our results in the form of the second equation.

2) The degree of the remainder is less than the degree of the divisor. This is very similar to the situation with ordinary division of positive integers, where the remainder is always less than the divisor.

The Division Algorithm

(3.2)

Let $f(x)$ and $p(x)$ be polynomials with $p(x)$ of lower degree than $f(x)$ and assume that $p(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = p(x) \cdot q(x) + r(x)$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $p(x)$.
The polynomials $f(x)$ and $p(x)$ are called the **dividend** and **divisor**, respectively, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

When $r(x) = 0$, we have $f(x) = g(x) \cdot q(x)$ and we say that $g(x)$ and $q(x)$ are **factors** of $f(x)$.

Exercise #4 Using long division to find a quotient and a remainder.
Divide $x^3 + 2x^2 - 4$ by $x-3$.

Synthetic Division (3.2)

- Synthetic division is a quick method of dividing polynomials.
- It can be used **when the divisor is of the form** $x-k$**.**
- In the synthetic division we write down only the essential parts of the long division table (the coefficients).

Exercise #5 (3.2 - #21)

Use synthetic division to perform the following divisions:

a) $\dfrac{2x^3 - 3x^2 + 4x - 5}{x - 2}$

b) If $f(x) = 2x^3 - 3x^2 + 4x - 5$, evaluate $f(2)$. What do you observe?

The Remainder Theorem (3.2)

> When we divide a polynomial $f(x)$ by $x - c$, the remainder is $f(c)$.

Proof

Exercise #6 (3.2 - #11)

Using the remainder theorem to evaluate a function and check for a factor.

Let $f(x) = x^4 - 6x^2 + 4x - 8$.

i) Evaluate $f(-3)$.

ii) Is $x + 3$ a factor of $f(x) = x^2 + 5x + 6$?

The Factor Theorem (3.2)

> The polynomial $x - c$ is a factor of the polynomial $f(x)$ if and only if $f(c) = 0$.

Proof

Exercise #7 Let $f(x)=2x^3-4x^2+2x-1$.

a) What is the remainder when dividing the given polynomial by $x-2$? In how many ways can you find the remainder?

b) Is $x-2$ a factor of $f(x)$?

c) Is $x-1$ a factor of $f(x)$?

Exercise #8 Find all values of k such that $f(x)$ is divisible by the given linear polynomial.

(3.2 - #39) $$f(x)=kx^3+x^2+k^2x+3k^2+11;\quad x+2$$

Exercise #9 Show that $x-c$ is not a factor of $f(x)$ for any real number c.

(3.2 - # 41) $$f(x)=3x^4+x^2+5$$

Exercise #10 Let $P(x,y)$ be a first – quadrant point on $y=6-x$ and consider the vertical line PQ with Q on the *x*-axis.

(3.2 - #47)

a) If PQ is rotated about the *y*-axis, determine the volume *V* of the resulting cylinder.

b) For what point $P(x,y)$ with $x\neq 1$ is the volume *V* in part (a) the same as the volume of the cylinder of radius 1 and altitude 5?

The Fundamental Theorem of Algebra

(3.3)

Every polynomial equation of degree more than or equal to 1 and complex coefficients

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0=0 \quad (n\geq 1, a_n\neq 0)$$

has at least one complex zero.(This zero may be a real number).

The Linear Factors Theorem

(Complete Factorization Theorem for Polynomials)

(3.3)

Every polynomial of degree $n>0$ can be expressed as a product of n linear factors.

$$f(x)=a_n(x-x_1)(x-x_2)...(x-x_n),$$

where a_n is the leading coefficient and x_i are the zeros of the polynomial.

Theorem

(3.3)

Every polynomial of degree $n\geq 1$ has exactly n roots, where a root of multiplicity k is counted k times.

Exercise #11 Write each polynomial as a product of linear factors.

a) $f(x) = 3x^2 - 5x - 2$

b) $f(x) = x^2 - 5$

c) $f(x) = x^2 - 4x + 5$

Exercise #12 Find the zeros of $f(x)$, then express $f(x)$ as a product of linear factors.

(3.3 - #17, #22) a) $f(x) = 4x^5 + 12x^4 + 9x^3$

b) $f(x) = x^4 + 21x^2 - 100$

Exercise #13 Factoring a polynomial given a zero.

a) Let $f(x) = 6x^3 + 13x^2 - 14x + 3$. Show that -3 is a zero and use this fact to factor $f(x)$ completely.

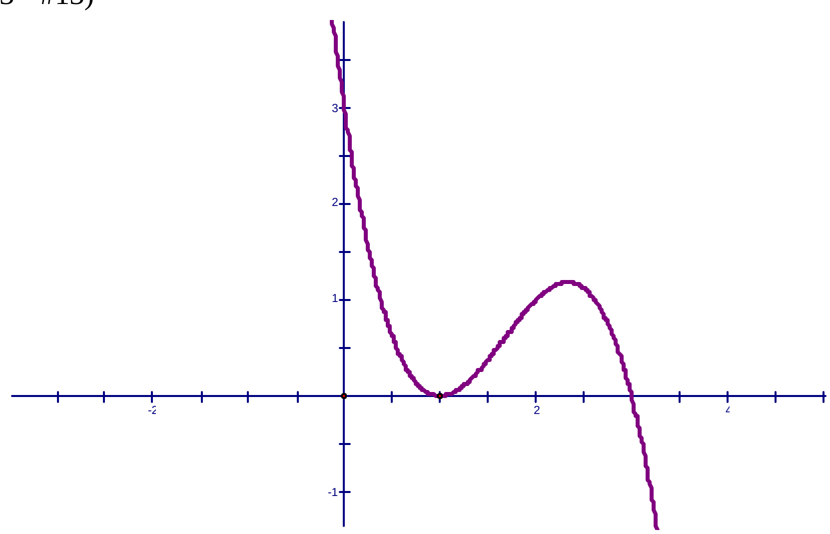
(3.3 – #24) b) $f(x) = x^4 - 9x^3 + 22x^2 - 32$. Knowing that 4 is a zero of multiplicity 4, factor $f(x)$ into linear factors.

Exercise #14 Finding polynomial equations satisfying given conditions.

(3.3 - #1) Find a polynomial function of degree 3 having the numbers -1, 2, and 3 as zeros and satisfying $f(-2) = 80$.

Exercise #15 Find a polynomial $f(x)$ of degree 4 with leading coefficient 1 such that both -5 and 2 are
(3.3 - # 7) zeros of multiplicity 2, and sketch the graph of *f*.

Exercise #16 Find the polynomial function of degree 3 whose graph is shown in the figure.
(3.3 - #13)



The Number and Location of Real Zeros

Descartes' Rule of Signs

In some cases, the following rule – discovered by the French philosopher and mathematician Rene Descartes around 1637 – is helpful in eliminating candidates from lengthy lists of possible rational roots.

To describe this rule, we need the concept of **variation in sign**. If $f(x)$ is a polynomial with real coefficient, written with descending powers of x (and omitting powers with coefficient 0), then **a variation in sign is a change from positive to negative or negative to positive in successive terms of the polynomial (adjacent coefficients have opposite signs).**

Example How many variations in sign occur in the following polynomial?

$$f(x) = 5x^7 - 3x^5 - x^4 + 2x^2 + x - 3$$

Descartes'Rule of Signs
(3.3)

> Let $f(x)$ be a polynomial with **real coefficients** and a **nonzero constant term.**
>
> a) The number of positive real zeros of $f(x)$ is either equal to the number of variations in sign in $f(x)$ or is less than that by an even whole number.
>
> b) The number of negative real zeros of $f(x)$ is either equal to the number of variations in sign in $f(-x)$ or is less than that by an even whole number.

Exercise #17 Use Descartes' rule of signs to determine the possible number of positive real zeros and
(3.3 - #27, #31) negative real zeros for each function, as well as the number of nonreal complex solutions.

a) $f(x) = 4x^3 - 6x^2 + x - 3$

b) $f(x) = 3x^4 + 2x^3 - 4x + 2$

First Theorem on Bounds for Real Zeros of Polynomials
(3.3)

> Suppose that $f(x)$ is a polynomial with **real coefficients** and a **positive leading coefficient** and that $f(x)$ is divided synthetically by $x - c$.
>
> 1) If $c > 0$ and if all numbers in the last row of the division process are **either positive or zero**, then c is an **upper bound** for the real zeros of $f(x)$.
>
> 2) If $c < 0$ and if the numbers in the last row of the division process are **alternately positive and negative** (and a 0 is considered to be either positive or negative), then c is a **lower bound** for the real zeros of $f(x)$

Exercise #18 (3.3 - #36)

Determine the smallest and largest integers that are upper and lower bounds, respectively, for the real solutions of the equation.

$$2x^3 - 5x^2 + 4x - 8 = 0$$

When a graphing utility is used, the following theorem is helpful in finding a viewing rectangle that shows all the zeros of a polynomial.

Second Theorem on Bounds for Real Zeros of Polynomials
(3.3)

Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ is a polynomial with eal coefficients. All of the real zeros of $f(x)$ are in the interval

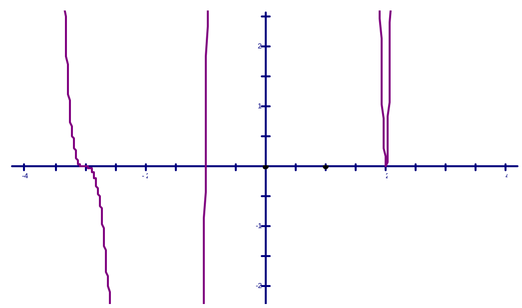
$$(-M, M),$$

where M is the ratio of the largest coefficient (in magnitude) to the absolute value of the leading coefficient, plus 1.

$$M = \frac{\max\left(|a_n|, |a_{n-1}|, ...|a_1|, |a_0|\right)}{|a_n|} + 1$$

Exercise #19 (3.3 - #43)

a) Find a factored form for a polynomial function that has minimal degree.
Assume that the intercept values are integers.
b) If the leading coefficient is 2, find the y-intercept.



Exercise #20 (3.3 - #51)

A scientist has limited data on the temperature T (in degrees Celsius) during a 24-hour period. If t denotes time in hours and t=0 corresponds to midnight, find the fourth-degree polynomial that fits the information in the following table.

t (hours)	0	5	12	19	24
T ($^\circ C$)	0	0	10	0	0

The Conjugate Zeros Theorem
(3.4)

> If $f(x)$ is a polynomial function with real coefficients and if $a+bi$ is a zero of $f(x)$, then its conjugate $a-bi$ is also a zero of $f(x)$.

Exercise #21 For the given polynomial , one zero is given. Find all the others.

$f(x)=x^3-7x^2+17x-15$; $2-i$

Finding all the rational zeros of a polynomial (3.4)

The Factor Theorem tells us that finding the zeros of a polynomial is really the same thing as factoring it into linear factors. We now study a method for finding all the rational zeros of a polynomial.

Example Consider the polynomial

$f(x)=(x-2)(x-3)(x+4)$ Factored form

$=$ ________________________________ Expanded form.

What are the zeros of $f(x)$? ____________________

What relationship exists between the zeros and the constant term of the polynomial?

____

The next theorem generalizes this observation.

The Rational Zeros Theorem
(3.4)

> If the polynomial $f(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$ ($a_0 \neq 0, a_n \neq 0$) has integer coefficients, then every **rational zero** of $f(x)$ is of the form $\frac{p}{q}$ where p is a factor of the constant coefficient a_0
> q is a factor of the leading coefficient a_n.

Note: The Rational Zeros Theorem gives only POSSIBLE rational zeros. It does not tell us whether these rational numbers are actual zeros.

Exercise #22 Using the Rational Zeros Theorem

Do each of the following for the polynomial function defined by

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2.$$

a) List all possible rational zeros.

b) Find all rational zeros and factor $f(x)$ into linear factors.

Finding the Rational Zeros of a Polynomial

1. List all possible rational zeros using the Rational Zeros Theorem.
2. Use synthetic division to evaluate the polynomial at each of the candidates for rational zeros that you found in Step 1. when the remainder is 0, note the quotient you have obtained.
3. Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is a quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

Exercise #23 For the given polynomial functions, do the following:

i) List the maximum number of real zeros;
ii) List the number of positive real zeros and negative real zeros;
iii) list all possible rational zeros;
iv) find all rational zeros;
v) factor $f(x)$.
vi) Graph the function.

a) $f(x) = x^3 + 6x^2 - x - 30$.

(3.4 - #21) b) $f(x) = 6x^5 + 19x^4 + x^3 - 6x^2$.

Exercise #24 (3.4 - #37) A storage shelter is to be constructed in the shape of a cube with a triangular prism forming the roof. The length x of a side of the cube is yet to be determined.
a) If the total height of the structure is 6 feet, find a formula for the volume in terms of x.
b) Determine x so that the volume is 80 cubic feet.

Exercise #25 a) Find all the complex zeros of $f(x) = x^4 - 6x^3 + 22x^2 - 30x + 13$.

b) Find all the solutions of $x^4 - 5x^3 - 5x^2 + 23x + 10 = 0$